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KEY

TO

THE NORMAL

ELEMENTARY ALGEBRA.

BY

EDWARD BROOKS, A.M.,

PRINCIPAL OF PENNSYLVANIA STATE NORMAL SCHOOL, AND AUTHOR OF "NORMAL SERIES OF ARITHMETICS," "NORMAL ALGEBRA," "NORMAL GEOMETRY," "PHILOSOPHY OF ARITHMETIC," ETC.

It is better to know much of a few things than to know a little of many things.

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KEY

TO

NORMAL ELEMENTARY ALGEBRA.

INTRODUCTION.

LESSON II.

7. Let x = cost of carriage;
$3x$ = cost of horses.

$x+3x=1000$;
$4x=1000$.
x = \$250, cost of carriage;
$3x$ = \$750, cost of horses.

8. Let x = what he saved;
$5x$ = what he spent.

$x+5x=1500$;
$6x=1500$;
x = \$250, what he saved.

LESSON III.

7. Let x = cost of lot;
$5x$ = cost of house.

$5x-x=2560$;
$4x=2560$.
x = \$640, cost of lot;
$5x$ = \$3200, cost of house.

8. Let x = B's gain;
$6x$ = A's gain.

$6x-x=650$;
$5x=650$.
x = \$130, B's gain;
$6x$ = \$780, A's gain.

LESSON IV.

7. Let x = my age.

$$\frac{x}{2}+\frac{x}{3}=40;$$
$$\frac{5x}{6}=40;$$
$$\frac{x}{6}=8;$$
$x=48$ years, my age.

8. Let x = number of miles.

$$4x-2\tfrac{1}{2}x=102;$$
$$\frac{3x}{2}=102;$$
$$\frac{x}{2}=34.$$
$x=68$, the number of miles.

9. Let x = Benton's money.

$$x-\frac{4x}{5}=\frac{x}{5};$$
$$\tfrac{3}{4}\text{ of }\frac{4x}{5}=\frac{3x}{5}.$$
$$\frac{x}{5}+\frac{3x}{5}=120;$$
$$\frac{4x}{5}=120;$$
$$\frac{x}{5}=30;$$
$x=\$150$, Benton's money.

10. Let x = Bessie's money;

$$x-\frac{3x}{4}=\frac{x}{4};$$
$$\tfrac{2}{3}\text{ of }\frac{3x}{4}=\frac{2x}{4}.$$
$$\frac{x}{4}+\frac{2x}{4}=30;$$
$$\frac{3x}{4}=30;$$
$$\frac{x}{4}=10;$$
$x=\$40$, Bessie's money.

LESSON V.

5. Let x = man's wages.

$\frac{x}{2}$ = wife's wages;

$\frac{x}{3}$ = son's wages.

$$x+\frac{x}{2}+\frac{x}{3}=22;$$
$$\frac{11x}{6}=22;$$
$$\frac{x}{6}=2.$$
$x=\$12$, man's wages;

$\frac{x}{2}=\$6$, wife's wages;

$\frac{x}{3}=\$4$, son's wages.

6. Let x = cost of sheep;

$5x$ = cost of cow;

$15x$ = cost of horse.

$$x+5x+15x=\$315;$$
$$21x=315.$$
$x=\$15$, cost of sheep;

$5x=\$75$, cost of cow;

$15x=\$225$, cost of horse.

7. Let x = C's tax;

$\frac{3x}{4}$ = B's tax;

$\frac{2}{3}$ of $\frac{3x}{4}=\frac{2x}{4}$ = A's tax.

$$x+\frac{3x}{4}+\frac{2x}{4}=450;$$

$$\frac{9x}{4}=450;$$
$$\frac{x}{4}=50.$$
$$x=\$200, \text{ C's tax};$$
$$\frac{3x}{4}=\$150, \text{ B's tax};$$
$$\frac{2x}{4}=\$100, \text{ A's tax}.$$

LESSON VI.

6. Let $x=$ my fortune.

$$\frac{x}{2}+\frac{x}{3}+380=2580;$$
$$\frac{5x}{6}+380=2580;$$
$$\frac{5x}{6}=2200;$$
$$\frac{x}{6}=440;$$
$$x=\$2640, \text{ my fortune.}$$

7. Let $x=$ the number of trees.

$$x-\frac{x}{3}-\frac{x}{4}=100;$$
$$\frac{5x}{12}=100;$$
$$\frac{x}{12}=20;$$
$$x=240, \text{ the number of trees.}$$

8. Let $x=$ the sum spent.

$$4x-\frac{3x}{5}-680=5100;$$
$$\frac{17x}{5}-680=5100;$$
$$\frac{17x}{5}=5780;$$
$$\frac{x}{5}=340;$$
$$x=\$1700, \text{ the sum spent.}$$

LESSON VII.

7. Let $x=$ what C contributed;
$2x-10=$ what A contributed;
$2x+10=$ what B contributed.

$$x+2x-10+2x+10=125;$$
$$5x=125.$$
$$x=\$25, \text{ C's contribution};$$
$$2x-10=\$40, \text{ A's contribution};$$
$$2x+10=\$60, \text{ B's contribution.}$$

8. Let $x=$ cost of hat;
$2x+4=$ cost of cloak;
$4x+8-4=$ cost of shawl.

$$x+2x+4+4x+8-4=78;$$
$$7x+8=78;$$
$$7x=70.$$
$$x=\$10, \text{ cost of hat};$$
$$2x+4=\$24, \text{ cost of cloak};$$
$$4x+4=\$44, \text{ cost of shawl.}$$

LESSON VIII.

5. Let $x = $ cost of cloak;

$\frac{2x}{3} + 4 = $ cost of shawl.

$$\frac{2x}{3} + 4 + 4 + 4 = x;$$

$$12 = \frac{x}{3}:$$

$$\frac{x}{3} = 12.$$

$x = \$36$, cost of cloak;

$\frac{2x}{3} + 4 = \$28$, cost of shawl.

6. Let $x = $ A's money;

$x + 12 = $ B's money;

$x + 12 + 6 = $ C's money.

$$x + x + 12 + x + 12 + 6 = 4\tfrac{1}{2}x;$$

$$3x + 30 = 4\tfrac{1}{2}x;$$

$$30 = 1\tfrac{1}{2}x;$$

$$\frac{3x}{2} = 30;$$

$$\frac{x}{2} = 10.$$

$x = \$20$, A's money;

$x + 12 = \$32$, B's money;

$x + 12 + 6 = \$38$, C's money.

7. Let $x = $ what it cost.

$$3x + 50 = 2(x + 150);$$

$$3x + 50 = 2x + 300.$$

$x = \$250$, what it cost.

8. Let $x = $ number of geese.

$$x + x + \frac{x}{2} + 2\tfrac{1}{2} = 100;$$

$$\frac{5x}{2} = \frac{195}{2};$$

$$\frac{x}{2} = \frac{39}{2};$$

$x = 39$, number of geese.

LESSON IX.

2. Let $x = $ Sarah's age;

$2x = $ Mary's age.

$$x + 2x = a;$$

$$3x = a.$$

$x = \frac{a}{3}$, Sarah's age;

$2x = \frac{2a}{3}$, Mary's age.

3. $\frac{a}{3} = \frac{36}{3}$, or 12;

$\frac{2a}{3} = \frac{72}{3}$, or 24.

4. Let $x = $ the smaller part;

$5x = $ the larger part.

$$x + 5x = m;$$

$$6x = m.$$

$x = \frac{m}{6}$, the smaller part;

$5x = \frac{5m}{6}$, the larger part.

6. Let $x = $ the smaller;

$5x = $ the larger.

$$5x - x = a;$$

$$4x = a.$$

$x = \frac{a}{4}$, the smaller;

$5x = \frac{5a}{4}$, the larger.

8. Let x = the number.

$$x+\frac{x}{3}=b;$$
$$\frac{4x}{3}=b;$$
$$\frac{x}{3}=\frac{b}{4}.$$
$$x=\frac{3b}{4}, \text{ the number.}$$

9. Let x = the third part;

$2x$ = the second part;

$4x$ = the first part.

$$x+2x+4x=c;$$
$$7x=c;$$
$$x=\frac{c}{7}, \text{ the third part;}$$
$$2x=\frac{2c}{7}, \text{ the second part;}$$
$$4x=\frac{4c}{7}, \text{ the first part.}$$

10. Let x = the number.

$$3x+n=a;$$
$$3x=a-n.$$
$$x=\frac{a-n}{3}, \text{ the number.}$$

11. Let x = larger number;

$x-c$ = smaller number.

$$x+x-c=a;$$
$$2x=a+c;$$
$$x=\frac{a+c}{2};$$
$$x-c=\frac{a+c}{2}-\frac{2c}{2}=\frac{a-c}{2}.$$

12. Let x = the length of pole.

$$\frac{x}{2}+\frac{x}{3}+h=x;$$
$$h=\frac{x}{6};$$
$$x=6h.$$

ADDITION.

Art. 59. (page 30.)

6. $-130xy$.

7. $15x^2y^3z$.

Art. 60. (page 31.)

8.
$$\begin{array}{r}
3ax-\ \ 2b^2c\qquad\quad \\
5ax\qquad\qquad+\ \ 7c^3 \\
9b^2c-12c^3 \\
8ax\qquad\qquad+15c^3 \\
14b^2c-18c^3 \\
\hline
16ax+21b^2c-\ \ 8c^3
\end{array}$$

9.
$$\begin{array}{r}
m+\ \ 3n^2-\ \ 5mn \\
3m-\ \ 8n^2\qquad\quad \\
7n^2-\ \ 8mn \\
19m\qquad\qquad+27mn \\
16n^2-17mn \\
\hline
23m+18n^2-\ \ 3mn
\end{array}$$

10.
$$\begin{array}{r}
a+2b+3c \\
2a-\ \ b-2c \\
-a+\ \ b-\ \ c \\
-a-\ \ b+\ \ c \\
\hline
a+\ \ b+\ \ c
\end{array}$$

11.
$$\begin{array}{r}
3a-\ \ 4p+\ \ \ \ q\qquad \\
7p+\ \ 3q-\ \ 6 \\
9a+\ \ 3p\qquad\qquad-\ \ 7 \\
+11p+\ \ 9q-12 \\
\hline
12a+17p+13q-25
\end{array}$$

12.
$$\begin{array}{r}
a+b-c\\
a-b+c\\
a+b+c\\
-a+b+c\\
\hline
2a+2b+2c
\end{array}$$

13.
$$\begin{array}{l}
4a+7a^2c-8m^3\\
7a \qquad\qquad +16m^3\\
\qquad 15a^2c-20m^3+17\\
\quad -22a^2c+12m^3-5\\
\hline
11a \qquad\qquad\qquad +12
\end{array}$$

14.
$$\begin{array}{r}
34ax^3-16ay^2 \qquad\qquad\qquad\\
-25ax^3+14ay^2-13ay^3 \qquad\\
+15ay^3+16\\
15ay^2 \qquad\qquad -16\\
22ax^3+7ay^2-11ay^3 \qquad\\
\hline
31ax^3+20ay^2-9ay^3 \qquad
\end{array}$$

15.
$$\begin{array}{r}
12x+9y-6z\\
5a+13x-12y \qquad\\
-16x+7y+10z\\
-5a+10x \qquad +12z\\
\hline
19x+4y+16z
\end{array}$$

16.
$$\begin{array}{r}
x^n-ax^2+3b \qquad\\
3ax^2-2b+y^{2n}\\
5x^n \qquad +4b-3y^{2n}\\
-4x^n-3ax^2+7b+y^{2n}\\
\hline
2x^n-ax^2+12b-y^{2n}
\end{array}$$

17.
$$\begin{array}{r}
5a-9b+5c+3-d\\
a-3b \qquad -8-d\\
3a+2b-3c+4+5d\\
2a \qquad +5c-6-3d\\
\hline
11a-10b+7c-7 \qquad
\end{array}$$

18.
$$\begin{array}{r}
x^4-4x^3y+6x^2y^2-4xy^3+y^4\\
4x^3y-12x^2y^2+12xy^3-4y^4\\
6x^2y^2-12xy^3+6y^4\\
4xy^3-4y^4\\
\hline
x^4 \qquad\qquad\qquad\qquad\qquad -y^4
\end{array}$$

19.
$$\begin{array}{l}
a^3+ab^2+ac^2-a^2b-abc-a^2c\\
\quad -ab^2 \qquad +a^2b-abc \qquad +b^3+bc^2-b^2c\\
\qquad\quad -ac^2 \qquad -abc+a^2c \qquad -bc^2+b^2c+c^3\\
\hline
a^3 \qquad\qquad\qquad -3abc \qquad +b^3 \qquad\qquad +c^3
\end{array}$$

20.
$$\begin{array}{l}
4ab-3mn+10am-6an\\
\qquad 7mn-7am+4an\\
3ab \qquad\qquad\qquad +7an+9\\
\quad -4mn-3am \qquad\quad +6-5n^2\\
\qquad\qquad\qquad\qquad -15+4n^2-2m^2\\
\hline
7ab \qquad\qquad\qquad +5an \qquad -n^2-2m^2
\end{array}$$

Art. 61. (page 32.)

8.
$$\begin{array}{r}
4ax+7(a^2-b^2)\\
6ax-5(a^2-b^2)\\
-5ax+3(a^2-b^2)\\
-7ax+12(a^2-b^2)\\
9ax+16(a^2-b^2)\\
-33(a^2-b^2)\\
\hline
7ax\qquad\qquad\qquad
\end{array}$$

9.
$$\begin{array}{lr}
2a^2-3(a+x) & \\
5a^2 & +6(x-y)^2\\
4a^2 & -7(x-y)^2\\
-6a^2+9(a+x) & \\
\quad 3(a+x) & -9(x-y)^2\\
a^2-(a+x) & +(x-y)^2\\
\hline
6a^2+8(a+x) & -9(x-y)^2
\end{array}$$

Art. 62. (page 33.)

4. $2ax-2bx=2(a-b)x$

$2(a-b)x+(a-b)x=3(a-b)x$

5.
$$\begin{array}{r}
4ax+3x\qquad\\
2ax-5x+bx\\
-5ax+2x-2bx\\
\hline
ax-bx=(a-b)x
\end{array}$$

6.
$$\begin{array}{l}
(3a-2b)y\\
(a+2b+c)y\\
\hline
(4a+c)y
\end{array}$$

7.
$$\begin{array}{l}
3an-5am\\
2an+3am-3bn-5m\\
\qquad+2am+6bn\\
\qquad\qquad-3bn+5m+cn\\
\hline
5an+cn=(5a+c)n
\end{array}$$

SUBTRACTION.

Art. 68. (page 36.)

16.
$$\begin{array}{l}
4a^m-3b^n\\
2a^m-5b^n\\
\hline
2a^m+2b^n
\end{array}$$

17.
$$\begin{array}{c}
a^2+2ab+b^2\\
a^2-2ab+b^2\\
\hline
4ab
\end{array}$$

18.
$$\begin{array}{r}
a^2\qquad-b^2\\
a^2-2ab+b^2\\
\hline
2ab-2b^2
\end{array}$$

19.
$$\begin{array}{r}
3a+c+d-f-8\\
3a+c-d\qquad\quad\\
\hline
2d-f-8
\end{array}$$

20.
$$\begin{array}{r}
4ab+3b^2-2c\qquad\\
4ab-2b^2\qquad-3d\\
\hline
5b^2-2c+3d
\end{array}$$

21.
$$\begin{array}{r}
7am-3bc-c^2\qquad\\
5am-3bc-2c^2-5x^3\\
\hline
2am\qquad+c^2+5x^3
\end{array}$$

22.
$$\begin{array}{r}
2a+2b-3c-8\\
-3a+4b+3c-5\\
\hline
5a-2b-6c-3
\end{array}$$

23.
$$\begin{array}{r}
a^3+3a^2b+3ab^2+b^3\\
a^3-3a^2b+3ab^2-b^3\\
\hline
6a^2b\qquad\qquad+2b^3
\end{array}$$

$$
\begin{array}{r}
\textbf{24.}\quad a^2-3ab-\ b^2+\ bc-2c^2\\
a^2-5ab-3b^2+5bc-2c^2\\
\hline
2ab+2b^2-4bc
\end{array}
$$

Art. 70. (page 38.)

9.
$$
\begin{array}{r}
4n^2c+3c\\
7c-4ac\\
\hline
4n^2c-4c+4ac=(n^2-1+a)4c
\end{array}
$$

10.
$$
\begin{array}{r}
an+cn+dn\\
an+\ n+dn\\
\hline
cn-n=(c-1)n
\end{array}
$$

11.
$$
\begin{array}{r}
(6a+2x)cd\\
(4a+2x)cd\\
\hline
2acd
\end{array}
$$

12.
$$
\begin{array}{r}
5a^2+10b^2\\
-3a^2+\ 2b^2\\
\hline
8a^2+\ 8b^2=8(a^2+b^2)
\end{array}
$$

13.
$$
\begin{array}{r}
6ay-3my\\
-5my+6cy\\
\hline
6ay+2my-6cy=2y(3a+m-3c)
\end{array}
$$

14.
$$
\begin{array}{r}
6\sqrt{c}-\ a\sqrt{c}+b\sqrt{c}\\
-2\sqrt{c}+2a\sqrt{c}+b\sqrt{c}-ax\sqrt{c}\\
\hline
8\sqrt{c}-3a\sqrt{c}+ax\sqrt{c}=(8-3a+ax)\sqrt{c}
\end{array}
$$

Art. 72. (page 39.)

3. $a-\{b-c-(d-e)\}=a-\{b-c-d+e\}=a-b+c+d-e.$

4. $2a-\{b-(a-2b)\}=2a-\{b-a+2b\}=2a-b+a-2b=3a-3b.$

5. $3a-\{b+(2a-b)-(a-b)\}=3a-\{b+2a-b-a+b\}=3a-\{a+b\}=2a-b.$

6. $7a-[3a-\{4a-(5a-2a)\}]=7a-[3a-\{4a-5a+2a\}]=7a-[3a-a]=5a.$

7. $6a-[4b-\{4a-(6a-4b)\}]=6a-[4b-\{4a-6a+4b\}]=6a-[4b-4a+6a-4b]=6a-4b+4a-6a+4b=4a.$

8. $a-[2b+\{3c-3a-(a+b)\}+\{2a-(b+c)\}]=a-[2b+\{3c-3a-a-b\}+\{2a-b-c\}]=a-[2b+3c-3a-a-b+2a-b-c]=3a-2c.$

MULTIPLICATION.

Art. 80. (page 44.)

13.
$$\begin{array}{l}
x^4-x^3z+x^2z^2-xz^3+z^4\\
x+z\\
\hline
x^5-x^4z+x^3z^2-x^2z^3+xz^4\\
\qquad x^4z-x^3z^2+x^2z^3-xz^4+z^5\\
\hline
x^5 \qquad\qquad\qquad\qquad\qquad +z^5
\end{array}$$

14.
$$\begin{array}{l}
a^{n-2}-b^{n-2}\\
a^2+b^2\\
\hline
a^n-a^2b^{n-2}\\
\qquad a^{n-2}b^2-b^n\\
\hline
a^n+a^{n-2}b^2-a^2b^{n-2}-b^n
\end{array}$$

15.
$$\begin{array}{l}
a^2x^3+x^2y^3\\
a^2x^3-x^2y^3\\
\hline
a^4x^6+a^2x^5y^3\\
\qquad -a^2x^5y^3-x^4y^6\\
\hline
a^4x^6 \qquad\qquad -x^4y^6
\end{array}$$

16.
$$\begin{array}{l}
x^{\frac{1}{2}}-y^{\frac{1}{2}}\\
x^{\frac{1}{2}}+y^{\frac{1}{2}}\\
\hline
x-x^{\frac{1}{2}}y^{\frac{1}{2}}\\
\qquad x^{\frac{1}{2}}y^{\frac{1}{2}}-y\\
\hline
x \qquad\qquad -y
\end{array}$$

17.
$$\begin{array}{l}
3\tfrac{1}{2}a^2+5\tfrac{1}{2}c^3\\
2\ a^2+4\ c^3\\
\hline
7\ a^4+11a^2c^3\\
\qquad +14a^2c^3+22c^6\\
\hline
7\ a^4+25a^2c^3+22c^6
\end{array}$$

18.
$$\begin{array}{l}
c^2+cd-d^2\\
c-d\\
\hline
c^3+c^2d-cd^2\\
\quad -c^2d-cd^2+d^3\\
\hline
c^3 \qquad\quad -2cd^2+d^3
\end{array}$$

19.
$$\begin{array}{l}
a^2-3ab+4ab^2\\
a^2+3ab-4ab^2\\
\hline
a^4-3a^3b+4a^3b^2\\
\quad +3a^3b \qquad\qquad -9a^2b^2+12a^2b^3\\
\qquad\qquad -4a^3b^2 \qquad\qquad +12a^2b^3-16a^2b^4\\
\hline
a^4 \qquad\qquad\qquad\quad -9a^2b^2+24a^2b^3-16a^2b^4
\end{array}$$

20.
$$\begin{array}{l}
n^2+np+p^2\\
n^2-np+p^2\\
\hline
n^4+n^3p+n^2p^2\\
\quad -n^3p-n^2p^2-np^3\\
\qquad\qquad +n^2p^2+np^3+p^4\\
\hline
n^4 \qquad\quad +n^2p^2 \qquad\quad +p^4
\end{array}$$

21.
$$\begin{array}{l}
a^2+2ab+b^2\\
a^2-2ab+b^2\\
\hline
a^4+2a^3b+a^2b^2\\
\quad -2a^3b-4a^2b^2-2ab^3\\
\qquad\qquad +a^2b^2+2ab^3+b^4\\
\hline
a^4 \qquad\quad -2a^2b^2 \qquad\quad +b^4
\end{array}$$

22.

$$\begin{array}{l} a^m + b^n \\ a^m - b^n \\ \hline a^{2m} + a^m b^n \\ \qquad - a^m b^n - b^{2n} \\ \hline a^{2m} \qquad\quad - b^{2n} \end{array}$$

23.

$$\begin{array}{l} a^n - b^m \\ a^n - b^m \\ \hline a^{2n} - a^n b^m \\ \qquad - a^n b^m + b^{2m} \\ \hline a^{2n} - 2a^n b^m + b^{2m} \end{array}$$

24.

$$\begin{array}{l} m^3 + m^2 n + mn^2 + n^3 \\ m - n \\ \hline m^4 + m^3 n + m^2 n^2 + mn^3 \\ \qquad - m^3 n - m^2 n^2 - mn^3 - n^4 \\ \hline m^4 \qquad\qquad\qquad\qquad\qquad - n^4 \end{array}$$

25.

$$\begin{array}{l} a^3 + 3a^2 b + 3ab^2 + b^3 \\ a^3 - 3a^2 b + 3ab^2 - b^3 \\ \hline a^6 + 3a^5 b + 3a^4 b^2 + a^3 b^3 \\ \quad - 3a^5 b - 9a^4 b^2 - 9a^3 b^3 - 3a^2 b^4 \\ \qquad\qquad + 3a^4 b^2 + 9a^3 b^3 + 9a^2 b^4 + 3ab^5 \\ \qquad\qquad\qquad - a^3 b^3 - 3a^2 b^4 - 3ab^5 - b^6 \\ \hline a^6 \qquad - 3a^4 b^2 \qquad + 3a^2 b^4 \qquad - b^6 \end{array}$$

26.

$$\begin{array}{l} a^4 - a^3 + a^2 - a + 1 \\ a + 1 \\ \hline a^5 - a^4 + a^3 - a^2 + a \\ \qquad a^4 - a^3 + a^2 - a + 1 \\ \hline a^5 \qquad\qquad\qquad\qquad + 1 \end{array}$$

27.

$$\begin{array}{l} 1 + c + c^2 \\ 1 - c + c^2 \\ \hline 1 + c + c^2 \\ \quad - c - c^2 - c^3 \\ \qquad\quad c^2 + c^3 + c^4 \\ \hline 1 \qquad + c^2 \qquad + c^4 \end{array}$$

$$\begin{array}{l} 1 + c^2 + c^4 \\ 1 - c^2 \\ \hline 1 + c^2 + c^4 \\ \quad - c^2 - c^4 - c^6 \\ \hline \qquad\qquad - c^6 \end{array} \qquad \begin{array}{l} 1 + c \\ 1 - c \\ \hline 1 + c \\ \quad - c - c^2 \\ \hline 1 - c^2 \end{array}$$

This example is most briefly worked by multiplying the quantities in pairs, and then multiplying together their products.

Art. 81. (page 45.)

4.
$$\begin{array}{lll}
a-2 & & a-3 \\
a+2 & & a+3 \\
\hline
a^2-2a & & a^2-3a \\
\quad +2a-4 & & \quad +3a-9 \\
\hline
a^2 \quad -4 & & a^2 \quad -9 \\
a^2 \quad -9 & & \\
\hline
a^4 \quad -4a^2 & & \\
\quad -9a^2+36 & & \\
\hline
a^4 \quad -13a^2+36 & &
\end{array}$$

5.
$$\begin{array}{l}
a+b \\
a-b \\
\hline
a^2+ab \\
\quad -ab-b^2 \\
\hline
a^2 \quad -b^2 \\
a^2 \quad -b^2 \\
\hline
a^4 \quad -a^2b^2 \\
\quad -a^2b^2+b^4 \\
\hline
a^4 \quad -2a^2b^2+b^4
\end{array}$$

6.
$$\begin{array}{l}
1-a+a^2-a^3 \\
1+a \\
\hline
1-a+a^2-a^3 \\
\quad +a-a^2+a^3-a^4 \\
\hline
1-a^4 \\
1+a^4 \\
\hline
1-a^4 \\
\quad a^4-a^8 \\
\hline
1 \quad -a^8
\end{array}$$

7.
$$\begin{array}{l}
a^2+a+1 \\
a-1 \\
\hline
a^3+a^2+a \\
\quad -a^2-a-1 \\
\hline
a^3-1 \\
a^3-1 \\
\hline
a^6-a^3 \\
\quad -a^3+1 \\
\hline
a^6-2a^3+1
\end{array}$$

In these examples the work can be shortened by a little care in selecting the quantities which we first multiply together. Two quantities having the same terms connected by different signs will generally give a product of a small number of terms.

DIVISION.

Art. 90. (page 51.)

12.
$$\begin{array}{l|l}
m^3-n^3 & m^2+mn+n^2 \\
\cline{2-2}
 & m-n \\
m^3+m^2n+mn^2 & \\
\hline
-m^2n-mn^2-n^3 & \\
-m^2n-mn^2-n^3 & \\
\hline
\end{array}$$

13.
$$\begin{array}{l|l}
a^3-1 & a-1 \\
\cline{2-2}
a^3-a^2 & a^2+a+1 \\
\hline
\quad a^2-1 & \\
\quad a^2-a & \\
\hline
\qquad a-1 & \\
\qquad a-1 & \\
\hline
\end{array}$$

14.
$$\begin{array}{l|l}
8x^3-27y^3 & 2x-3y \\ \cline{2-2}
8x^3-12x^2y & 4x^2+6xy+9y^2 \\ \cline{1-1}
\quad 12x^2y-27y^3 & \\
\quad 12x^2y-18xy^2 & \\ \cline{1-1}
\qquad 18xy^2-27y^3 & \\
\qquad 18xy^2-27y^3 & \\ \cline{1-1}
\end{array}$$

15.
$$\begin{array}{l|l}
a^4-x^4 & a-x \\ \cline{2-2}
a^4-a^3x & a^3+a^2x+ax^2+x^3 \\ \cline{1-1}
\quad a^3x-x^4 & \\
\quad a^3x-a^2x^2 & \\ \cline{1-1}
\qquad a^2x^2-x^4 & \\
\qquad a^2x^2-ax^3 & \\ \cline{1-1}
\qquad\quad ax^3-x^4 & \\
\qquad\quad ax^3-x^4 & \\ \cline{1-1}
\end{array}$$

16.
$$\begin{array}{l|l}
a^4+2a^2b^2+9b^4 & a^2-2ab+3b^2 \\ \cline{2-2}
a^4-2a^3b+3a^2b^2 & a^2+2ab+3b^2 \\ \cline{1-1}
\quad 2a^3b-a^2b^2+9b^4 & \\
\quad 2a^3b-4a^2b^2+6ab^3 & \\ \cline{1-1}
\qquad 3a^2b^2-6ab^3+9b^4 & \\
\qquad 3a^2b^2-6ab^3+9b^4 & \\ \cline{1-1}
\end{array}$$

17.
$$\begin{array}{l|l}
a^4+a^2c^2+c^4 & a^2-ac+c^2 \\ \cline{2-2}
a^4-a^3c+a^2c^2 & a^2+ac+c^2 \\ \cline{1-1}
\quad a^3c+c^4 & \\
\quad a^3c-a^2c^2+ac^3 & \\ \cline{1-1}
\qquad a^2c^2-ac^3+c^4 & \\
\qquad a^2c^2-ac^3+c^4 & \\ \cline{1-1}
\end{array}$$

18.
$$\begin{array}{l|l}
x^2+2xy+y^2-z^2 & x+y-z \\ \cline{2-2}
x^2+xy-xz & x+y+z \\ \cline{1-1}
\quad xy+y^2+xz & \\
\quad xy+y^2-zy & \\ \cline{1-1}
\qquad xz+zy-z^2 & \\
\qquad xz+zy-z^2 & \\ \cline{1-1}
\end{array}$$

19.
$$\begin{array}{l|l}
m^4-n^4 & m^2+n^2 \\ \cline{2-2}
m^4+m^2n^2 & m^2-n^2 \\ \cline{1-1}
\quad -m^2n^2-n^4 & \\
\quad -m^2n^2-n^4 & \\ \cline{1-1}
\end{array}$$

20.
$$\begin{array}{l|l}
a^4-1 & a-1 \\ \cline{2-2}
a^4-a^3 & a^3+a^2+a+1 \\ \cline{1-1}
\quad a^3-1 & \\
\quad a^3-a^2 & \\ \cline{1-1}
\qquad a^2-1 & \\
\qquad a^2-a & \\ \cline{1-1}
\qquad\quad a-1 & \\
\qquad\quad a-1 & \\ \cline{1-1}
\end{array}$$

21.
$$\begin{array}{l|l}
a^{2n}-b^2 & a^n-b^n \\ \cline{2-2}
a^{2n}-a^nb^n & a^n+b^n \\ \cline{1-1}
\quad a^nb^n-b^{2n} & \\
\quad a^nb^n-b^{2n} & \\ \cline{1-1}
\end{array}$$

22.
$$\begin{array}{l|l}
m^5-n^5 & m-n \\ \hline
m^5-m^4n & m^4+m^3n+m^2n^2+mn^3+n^4
\end{array}$$
$$\begin{aligned}
&\underline{\quad\ m^4n-n^5} \\
&\quad\ m^4n-m^3n^2 \\
&\underline{\qquad\qquad m^3n^2-n^5} \\
&\qquad\qquad m^3n^2-m^2n^3 \\
&\underline{\qquad\qquad\qquad m^2n^3-n^5} \\
&\qquad\qquad\qquad m^2n^3-mn^4 \\
&\underline{\qquad\qquad\qquad\qquad mn^4-n^5} \\
&\qquad\qquad\qquad\qquad \underline{mn^4-n^5}
\end{aligned}$$

23.
$$\begin{array}{l|l}
s^4-t^4 & s^3+s^2t+st^2+t^3 \\ \hline
s^4+s^3t+s^2t^2+st^3 & s-t
\end{array}$$
$$\begin{aligned}
&-s^3t-s^2t^2-st^3-t^4 \\
&\underline{-s^3t-s^2t^2-st^3-t^4}
\end{aligned}$$

24.
$$\begin{array}{l|l}
a^{2n}+2a^nb^n+b^{2n} & a^n+b^n \\ \hline
a^{2n}+\ a^nb^n & a^n+b^n
\end{array}$$
$$\begin{aligned}
&a^nb^n+b^{2n} \\
&\underline{a^nb^n+b^{2n}}
\end{aligned}$$

25.
$$\begin{array}{l|l}
27x^3-64y^3 & 3x-4y \\ \hline
27x^3-36x^2y & 9x^2+12xy+16y^2
\end{array}$$
$$\begin{aligned}
&36x^2y-64y^3 \\
&36x^2y-48xy^2 \\
&\underline{\qquad\quad 48xy^2-64y^3} \\
&\qquad\quad \underline{48xy^2-64y^3}
\end{aligned}$$

26.
$$\begin{array}{l|l}
x^5+1 & x+1 \\ \hline
x^5+x^4 & x^4-x^3+x^2-x+1
\end{array}$$
$$\begin{aligned}
&-x^4+1 \\
&\underline{-x^4-x^3} \\
&\qquad x^3+1 \\
&\qquad \underline{x^3+x^2} \\
&\qquad\quad -x^2+1 \\
&\qquad\quad \underline{-x^2-x} \\
&\qquad\qquad\quad x+1 \\
&\qquad\qquad\quad \underline{x+1}
\end{aligned}$$

27.
$$\begin{array}{l|l}
1-z^5 & 1-z \\ \hline
1-z & 1+z+z^2+z^3+z^4
\end{array}$$
$$\begin{aligned}
&z-z^5 \\
&\underline{z-z^2} \\
&\quad z^2-z^5 \\
&\quad \underline{z^2-z^3} \\
&\qquad z^3-z^5 \\
&\qquad \underline{z^3-z^4} \\
&\qquad\quad z^4-z^5 \\
&\qquad\quad \underline{z^4-z^5}
\end{aligned}$$

28.
$$\begin{array}{l|l}
a^{3n}-b^{3n} & a^n-b^n \\ \hline
a^{3n}-a^{2n}b^n & a^{2n}+a^nb^n+b^{2n}
\end{array}$$
$$\begin{aligned}
&a^{2n}b^n-b^{3n} \\
&\underline{a^{2n}b^n-a^nb^{2n}} \\
&\qquad a^nb^{2n}-b^{3n} \\
&\qquad \underline{a^nb^{2n}-b^{3n}}
\end{aligned}$$

29.
$$\begin{array}{l|l}
a^6-b^6 & a^3+2a^2b+2ab^2+b^3 \\
\underline{a^6+2a^5b+2a^4b^2+a^3b^3} & a^3-2a^2b+2ab^2-b^3 \\
\quad -2a^5b-2a^4b^2-a^3b^3-b^6 & \\
\quad \underline{-2a^5b-4a^4b^2-4a^3b^3-2a^2b^4} & \\
\qquad 2a^4b^2+3a^3b^3+2a^2b^4-b^6 & \\
\qquad \underline{2a^4b^2+4a^3b^3+4a^2b^4+2ab^5} & \\
\qquad\quad -a^3b^3-2a^2b^4-2ab^5-b^6 & \\
\qquad\quad \underline{-a^3b^3-2a^2b^4-2ab^5-b^6} &
\end{array}$$

30.
$$\begin{array}{l|l}
(a-x)^2-(x-y)^2 & (a-x)-(x-y) \\
\underline{(a-x)^2-(a-x)(x-y)} & (a-x)+(x-y)=a-y \\
\quad a-x)(x-y)-(x-y)^2 & \\
\quad \underline{(a-x)(x-y)-(x-y)^2} &
\end{array}$$

Art. 94. (page 54.)

12.
$$\begin{array}{l|l}
a^{-3n}-b^{6n} & a^{-n}-b^{2n} \\
\underline{a^{-3n}-a^{-2n}b^{2n}} & a^{-2n}+a^{-n}b^{2n}+b^{4n} \\
\quad a^{-2n}b^{2n}-b^{6n} & \\
\quad \underline{a^{-2n}b^{2n}-a^{-n}b^{4n}} & \\
\qquad a^{-n}b^{4n}-b^{6n} & \\
\qquad \underline{a^{-n}b^{4n}-b^{6n}} &
\end{array}$$

COMPOSITION AND FACTORING.

Art. 98. (page 58.)

3. $(a+c)(a-c)=a^2-c^2$

$$\begin{array}{r}
a^2-c^2 \\
\underline{a+c} \\
a^3-ac^2 \\
\underline{+a^2c-c^3} \\
a^3+a^2c-ac^2-c^3
\end{array}$$

4. $(x-3)(x-3)=x^2-6x+9$

$$\begin{array}{r}
x^2-6x+9 \\
\underline{x-4} \\
x^3-6x^2+9x \\
\underline{-4x^2+24x-36} \\
x^3-10x^2+33x-36
\end{array}$$

5. $(1+x)(1-x)=1-x^2$

$$\begin{array}{l} 1-x^2 \\ 1-x \\ \hline 1-x^2 \\ \quad -x+x^3 \\ \hline 1-x-x^2+x^3 \end{array}$$

6. $(1+x)(1-x)=1-x^2$
$(1-x^2)(1-x^2)=1-2x^2+x^4$

7. $(x+y)(x-y)=x^2-y^2$
$(x^2-y^2)(x^2+y^2)=x^4-y^4$

11. $(a+\overline{b-c})(a-\overline{b-c})=a^2$
$-(b-c)^2=a^2-b^2+2bc-c^2$

13. $(a+c)(a-c)=a^2-c^2$
$(a^2-c^2)(a^2-c^2)=a^4-2a^2c^2+c^4$

14. $(x-3)(x-4)=x^2-7x+12$

$$\begin{array}{l} x^2-7x+12 \\ x+5 \\ \hline x^3-7x^2+12x \\ \quad +5x^2-35x+60 \\ \hline x^3-2x^2-23x+60 \end{array}$$

15. $(a+1)(a-1)=a^2-1$
$(a+2)(a-2)=a^2-4$
$(a^2-1)(a^2-4)=a^4-5a^2+4$

16. $(a^n-b^m)(a^n+b^m)=a^{2n}-b^{2m}$
$(a^{2n}-b^{2m})(a^{2n}+b^{2m})=a^{4n}-b^{4m}$

Art. 108. (page 62.)

9. $x^2y^2-y^2=y^2(x^2-1)=y^2(x+1)(x-1)$

10. $a^8-b^8=(a^4+b^4)(a^4-b^4)=(a^4+b^4)(a^2+b^2)(a^2-b^2)$
$=(a^4+b^4)(a^2+b^2)(a+b)(a-b).$

Art. 109. (page 63.)

8. $x^4-9x^2-36=(x-12)(x+3)$; because $-12+3=-9$, and $-12\times 3=-36$.

9. $4x^2-6x-40=(2x+5)(2x-8)$; because $(-8+5)2=-6$, and $-8\times 5=-40$.

10. $a^2+4ac-21c^2=(a+7c)(a-3c)$; because $+7-3=+4$, and $7\times -3=-21$.

11. $a^{2n}+5a^n-84=(a^n+12)(a^n-7)$; because $+12-7=+5$, and $12\times -7=-84$.

GREATEST COMMON DIVISOR.

Art. 115. (page 66.)

9. $a^4-b^4=(a^2+b^2)(a+b)(a-b)$
$a^6-b^6=(a^2-ab+b^2)(a^2+ab+b^2)(a-b)(a+b)$
G. C. D. $=(a+b)(a-b)=a^2-b^2$

10. $x^2-y^2=(x+y)(x-y)$
$ax+ay+bx+by=(x+y)(a+b)$
G. C. D. $=x+y$

11. $ac+bc+ad+bd=(a+b)(c+d)$
$ac+bc-ad-bd=(a+b)(c-d)$
G. C. D. $=a+b$

Art. 116. (page 69.)

4. Rejecting the factor x from the first quantity and 2 from the second, and dividing,

$$\begin{array}{l|l} 2x^2-5x+3 & 2x^2-x-1 \\ \underline{2x^2-\ \ x-1} & 1 \\ \quad -4x+4 & \end{array}$$

Rejecting the factor -4 from the remainder, and dividing the divisor by it,

$$\begin{array}{l|l} 2x^2-x-1 & x-1=\text{G. C. D.} \\ \underline{2x^2-2x} & -2x+1 \\ \qquad x-1 & \\ \qquad \underline{x-1} & \end{array}$$

5.
$$\begin{array}{l|l} a^3-x^3 & a^2-x^2 \\ \underline{a^3-ax^2} & a \\ \quad ax^2-x^3 & \end{array}$$

Rejecting the factor x^2 from the remainder, and dividing,

$$\begin{array}{l|l} a^2-x^2 & a-x=\text{G. C. D.} \\ \underline{a^2-ax} & a+x \\ \quad ax-x^2 & \\ \quad \underline{ax-x^2} & \end{array}$$

6. Rejecting the factor $b+d$ from the first quantity, and dividing,

$$\begin{array}{l|l} a^2-c^2 & a+c=\text{G. C. D.} \\ \underline{a^2+ac} & a-c \\ \quad -ac-c^2 & \\ \quad \underline{-ac-c^2} & \end{array}$$

7.
$$\begin{array}{l|l} a^3+x^3 & a^2-x^2 \\ \underline{a^3-ax^2} & a \\ \quad ax^2+x^3 & \end{array}$$

Rejecting the factor x^2 from the remainder, and dividing,

$$\begin{array}{l|l} a^2-x^2 & a+x=\text{G. C. D.} \\ \underline{a^2+ax} & a-x \\ \quad -ax-x^2 & \\ \quad \underline{-ax-x^2} & \end{array}$$

8. Setting aside the common factor $2a$, and dividing,

$$\begin{array}{l|l} 2x^3+2y^3 & x^2-y^2 \\ \hline 2x^3-2xy^2 & 2x \\ \hline 2xy^2+2y^3 & \end{array}$$

Omitting the factor $2y^2$ of the remainder, and dividing,

$$\begin{array}{l|l} x^2-y^2 & x+y \quad (x+y)2a=\text{G. C. D.} \\ \hline x^2+xy & x-y \\ \hline -xy-y^2 & \\ -xy-y^2 & \end{array}$$

9.
$$\begin{array}{l|l} a^4-b^4 & a^3+a^2b-ab^2-b^3 \\ \hline a^4+a^3b-a^2b^2-ab^3 & a-b \\ \hline -a^3b+a^2b^2+ab^3-b^4 & \\ -a^3b-a^2b^2+ab^3+b^4 & \\ \hline 2a^2b^2 \qquad -2b^4 & \end{array}$$

Rejecting the factor $2b^2$ from this remainder, and dividing,

$$\begin{array}{l|l} a^3+a^2b-ab^2-b^3 & a^2-b^2=\text{G. C. D.} \\ \hline a^3 \qquad -ab^2 & a+b \\ \hline a^2b \qquad -b^3 & \\ a^2b \qquad -b^3 & \end{array}$$

10. Rejecting the factor x from the first quantity, and dividing,

$$\begin{array}{l|l} x^2-4x-21 & x^2-x-12 \\ \hline x^2-\ x-12 & 1 \\ \hline -3x-\ 9 & \end{array}$$

Rejecting -3 from this remainder, and dividing,

$$\begin{array}{l|l} x^2-\ x-12 & x+3 \text{ G. C. D.} \\ \hline x^2+3x & x-4 \\ \hline -4x-12 & \\ -4x-12 & \end{array}$$

LEAST COMMON MULTIPLE.

Art. 120. (page 70.)

7.
$$\begin{array}{l} a^2-b^2 \qquad\quad =(a+b)(a-b) \\ a^2-2ab+b^2=(a-b)(a-b) \\ \hline \text{L. C. M.}=(a+b)(a-b)(a-b)=a^3-a^2b-ab^2+b^3 \end{array}$$

8.
$$\begin{array}{l} a^2(a-z)=a^2(a-z) \\ x^2(a^2-z^2)=x^2(a-z)(a+z) \\ \hline \text{L. C. M.}=a^2x^2(a-z)(a+z)=a^2x^2(a^2-z^2) \end{array}$$

9. $3x^2(2a-1)=3x^2(2a-1)$
$4xy(4a^2-1)=2\times 2xy(2a+1(2a-1)$

L. C. M. $=3\times 2\times 2x^2y(2a+1)(2a-1)$
$=12x^2y(4a^2-1)$

10. $x^2-y^2=(x+y)(x-y)$
$x^3-y^3=(x-y)(x^2+xy+y^2)$

L. C. M. $=(x+y)(x-y)(x^2+xy+y^2)$
$=x^4+x^3y-xy^3-y^4$

11. $3a(a-b)=3a(a-b)$
$4ac(a^2-b^2)=2\times 2ac(a+b)(a-b)$
$6c^2x(a+b)=3\times 2c^2x(a+b)$

L. C. M. $=3\times 2\times 2ac^2x(a+b)(a-b)$
$=12ac^2x(a^2-b^2)$

12. $m^2+2mn+n^2=(m+n)(m+n)$
$m^3+n^3=(m+n)(m^2-mn+n^2)$

L. C. M. $=(m+n)(m+n)(m^2-mn+n^2)$
$=(m+n)(m^3+n^3)$

Art. 121. (page 72.)

2.
$$\begin{array}{l|l} x^2-4x-21 & x^2-x-12 \\ x^2-x-12 & 1 \\ \hline -3x-9 & \end{array}$$

Omitting the factor -3, and dividing,

$$\begin{array}{l|l} x^2-x-12 & x+3 \text{ G. C. D.} \\ x^2+3x & x-4 \\ \hline -4x-12 & \\ -4x-12 & \\ \hline \end{array}$$

L.C.M. $=\dfrac{x^2-x-12}{x+3}\times x^2-4x-21$
$=x^3-8x^2-5x+84.$

3.
$$\begin{array}{l|l} x^2+6x+8 & x^2+5x+6 \\ x^2+5x+6 & 1 \\ \hline x+2 & \end{array}$$

$$\begin{array}{l|l} x^2+5x+6 & x+2=\text{G. C. D.} \\ x^2+2x & x+3 \\ \hline 3x+6 & \\ 3x+6 & \\ \hline \end{array}$$

L. C. M $=\dfrac{x^2+5x+6}{x+2}\times x^2+6x+8$
$=x^3+9x^2+26x+24.$

4. $$\begin{array}{r|l} a^2+4ab+3b^2 & a^2-b^2 \\ \underline{a^2 \qquad\quad -b^2} & 1 \\ 4ab+4b^2 & \end{array}$$

Omitting the factor $4b$, and dividing,

$$\begin{array}{r|l} a^2-b^2 & a+b \quad \text{G. C. D.} \\ \underline{a^2+ab} & a-b \\ -ab-b^2 & \\ \underline{-ab-b^2} & \end{array}$$

$$\text{L. C. M.}=\frac{a^2-b^2}{a+b}\times a^2+4ab+3b^2$$
$$=a^3+3a^2b-ab^2-3b^3$$

5. $$\begin{array}{r|l} a^2+3ab+2b^2 & a^2-ab-6b^2 \\ \underline{a^2-\ ab-6b^2} & 1 \\ 4ab+8b^2 & \end{array}$$

Omitting the factor $4b$, and dividing,

$$\begin{array}{r|l} a^2-ab-6b^2 & a+2b \quad \text{G. C. D.} \\ \underline{a^2+2ab} & a-3b \\ -3ab-6b^2 & \\ \underline{-3ab-6b^2} & \end{array}$$

$$\text{L.C.M.}=\frac{a^2-ab-6b^2}{a+2b}\times a^2+3ab+2b^2$$
$$=a^3-7ab^2-6b^3$$

6. $$\begin{array}{r|l} x^2-ax+3x-3a & x^2-ax-3x+3a \\ \underline{x^2-ax-3x+3a} & 1 \\ 6x-6a & \end{array}$$

Omitting the factor 6, and dividing,

$$\begin{array}{r|l} x^2-ax-3x+3a & x-a \quad \text{G. C. D.} \\ \underline{x^2-ax} & x-3 \\ -3x+3a & \\ \underline{-3x+3a} & \end{array}$$

$$\text{L. C. M.}=\frac{x^2-ax-3x+3a}{x-a}\times x^2-ax+3x-3a$$
$$=x^3-ax^2-9x+9a$$

7. $$\begin{array}{r|l} x^2+3x+2 & x^2-x-2 \\ \underline{x^2-\ x-2} & 1 \\ 4x+4 & \end{array}$$

Omitting the factor 4, and dividing,

$$\begin{array}{r|l} x^2-x-2 & x+1 \\ \underline{x^2+x} & x-2 \\ -2x-2 & \\ \underline{-2x-2} & \end{array}$$

$$\begin{array}{r|l} x^2+5x+4 & x+1=\text{G. C. D.} \\ \underline{x^2+\ x} & x+4 \\ 4x+4 & \\ \underline{4x+4} & \end{array}$$

$$\text{L. C. M.}=\frac{x^2-x-2}{x+1}\times\frac{x^2+5x+4}{x+1}\times x^2+3x+2$$
$$=x^4+5x^3-20x-16$$

FRACTIONS.

Art. 132. (page 76.)

NOTE.—In performing examples upon the blackboard when the number has been factored, the common factors should be canceled by drawing a line through them. This cannot be done in type, except in the case of figures, and therefore we are obliged to let the factors stand uncanceled.

6. $$\frac{2a+2b}{a^2-b^2}=\frac{2(a+b)}{(a+b)(a-b)}=\frac{2}{a-b}.$$

7. $$\frac{a^2-1}{2(ab-b)}=\frac{(a+1)(a-1)}{2b(a-1)}=\frac{a+1}{2b}.$$

10. $$\frac{x^2-9}{2x^2+10x+12}=\frac{(x+3)(x-3)}{(x+3)(2x+4)}=\frac{x-3}{2x+4}.$$

11. $$\frac{a^3-ab^2}{a^2+2ab+b^2}=\frac{a(a+b)(a-b)}{(a+b)(a+b)}=\frac{a(a-b)}{a+b}.$$

12. $$\frac{x^2-4a^2}{x^2+2ax-8a^2}=\frac{(x-2a)(x+2a)}{(x-2a)(x+4a)}=\frac{x+2a}{x+4a}.$$

13. $$\frac{x^{2n}-9b^{2n}}{x^{2n}-6b^nx^n+9b^{2n}}=\frac{(x^n-3b^n)(x^n+3b^n)}{(x^n-3b^n)(x^n-3b^n)}=\frac{x^n+3b^n}{x^n-3b^n}.$$

Art. 133. (page 77.)

NOTE.—These fractions should be reduced to their lowest terms before dividing.

3. $$\begin{array}{r|l} 2ax+x^2 & a+x \\ \underline{2ax+2x^2} & 2x-\dfrac{x^2}{a+x} \quad \textit{Ans.} \\ -x^2 & \end{array}$$

4. $$\begin{array}{r|l} a^2-4c^2 & a-c \\ \underline{a^2-ac} & a+c-\dfrac{3c^2}{a-c} \\ ac-4c^2 & \\ \underline{ac-c^2} & \\ -3c^2 & \end{array}$$

5. $$\begin{array}{r|l} 3a^3-3x^3 & a-x \\ \underline{3a^3-3a^2x} & 3a^2+3ax+3x^2=3(a^2+ax+x^2) \quad \textit{Ans.} \\ 3a^2x-3x^3 & \\ \underline{3a^2x-3ax^2} & \\ 3ax^2-3x^3 & \\ \underline{3ax^2-3x^3} & \end{array}$$

6. $\dfrac{x^3-z^3}{(x-z)^2}=\dfrac{x^2+xz+z^2}{x-z}$

$$\begin{array}{l|l}
x^2+xz+z^2 & x-z \\
\underline{x^2-xz} & x+2z+\dfrac{3z^2}{x-z} \quad \textit{Ans.} \\
\quad 2xz+z^2 & \\
\quad \underline{2xz-2z^2} & \\
\qquad 3z^2 &
\end{array}$$

7.
$$\begin{array}{l|l}
a^3-b^3 & a+b \\
\underline{a^3+a^2b} & a^2-ab+b^2-\dfrac{2b^3}{a+b} \quad \textit{Ans.} \\
\quad -a^2b-b^3 & \\
\quad \underline{-a^2b-ab^2} & \\
\qquad ab^2-b^3 & \\
\qquad \underline{ab^2+b^3} & \\
\qquad\quad -2b^3 &
\end{array}$$

8. $\dfrac{x^4-z^4}{(x^2-z^2)^2}=\dfrac{x^2+z^2}{x^2-z^2}=1+\dfrac{2z^2}{x^2-z^2}.$

9. $\dfrac{x^3+z^3}{(x+z)^3}=\dfrac{x^2-xz+z^2}{x^2+2xz+z^2}=1-\dfrac{3xz}{x^2+2xz+z^2}.$

Art. 134. (page 78.)

5. $a+c-\dfrac{3c^2}{a-c}=\dfrac{a^2-c^2-3c^2}{a-c}=\dfrac{a^2-4c^2}{a-c}.$

6. $4x-\dfrac{3-5x}{4}=\dfrac{16x-3+5x}{4}=\dfrac{21x-3}{4}.$

7. $a-\dfrac{2ac-c^2}{a}=\dfrac{a^2-2ac+c^2}{a}=\dfrac{(a-c)^2}{a}$

8. $4a+x+\dfrac{3ax+x^2}{a-x}=\dfrac{4a^2-3ax-x^2+3ax+x^2}{a-x}=\dfrac{4a^2}{a-x}.$

9. $2x-5-\dfrac{2x^2+4}{x-3}=\dfrac{2x^2-11x+15-2x^2-4}{x-3}=\dfrac{11(1-x)}{x-3}.$

10. $a+x-\dfrac{c^2-x^2}{a-x}=\dfrac{a^2-x^2-c^2+x^2}{a-x}=\dfrac{a^2-c^2}{a-x}.$

Art. 135. (page 79.)

8. $\dfrac{x^2-2xy+y^2}{x^2-y^2}=\dfrac{x-y}{x+y}=(x-y)(x+y)^{-1}.$

9. $\dfrac{a(b-c)}{(b+c)^{-1}}=a(b-c)(b+c)=a(b^2-c^2).$

10. $\dfrac{4a(c-z)^{-1}}{c+z}=\dfrac{4a}{(c-z)c+z}=\dfrac{4a}{c^2-z^2}.$

11. $\dfrac{(a-b)^2(x-y)^{-1}}{(a-b)^{-1}(x-y)}=\dfrac{(a-b)^2(a-b)}{(x-y)(x-y)}=\dfrac{(a-b)^3}{(x-y)^2}.$

Art. 138. (page 81.)

6. $\dfrac{a-b}{a^2c}$, $\dfrac{a+b}{3ac^2}$, $5\frac{1}{2}=\frac{11}{2}$; L. C. M. $=6a^2c^2$.

$6a^2c^2\div a^2c=6c$; $6a^2c^2\div 3ac^2=2a$; $6a^2c^2\div 2=3a^2c^2$.

$$\frac{a-b}{a^2c}=\frac{(a-b)\times 6c}{a^2c\times 6c}=\frac{6c(a-b)}{6a^2c^2};$$

$$\frac{a+b}{3ac^2}=\frac{(a+b)\times 2a}{3ac^2\times 2a}=\frac{2a(a+b)}{6a^2c^2};$$

$$\frac{11}{2}=\frac{11\times 3a^2c^2}{2\times 3a^2c^2}=\frac{33a^2c^2}{6a^2c^2}.$$

7. $\dfrac{ab}{a-b}$, $\dfrac{bc}{a+b}$, $\dfrac{cd}{a^2-b^2}$; L. C. M. $=a^2-b^2$.

$(a^2-b^2)\div(a-b)=a+b$; $(a^2-b^2)\div(a+b)=a-b$.

$$\frac{ab}{a-b}=\frac{ab\times(a+b)}{(a-b)(a+b)}=\frac{ab(a+b)}{a^2-b^2};$$

$$\frac{bc}{a+b}=\frac{bc\times(a-b)}{(a+b)(a-b)}=\frac{bc(a-b)}{a^2-b^2}.$$

$$\frac{cd}{a^2-b^2}=\frac{cd}{a^2-b^2}.$$

8. $\dfrac{2ax}{x-1}$, $\dfrac{3ax}{x+1}$, $\dfrac{4ax}{x^2-1}$; L. C. M. $=x^2-1$.

$(x^2-1)\div(x-1)=x+1$; $(x^2-1)\div x+1=x-1$.

$$\frac{2ax}{x-1}=\frac{2ax(x+1)}{(x-1)(x+1)}=\frac{2ax(x+1)}{x^2-1};$$

$$\frac{3ax}{x+1}=\frac{3ax(x-1)}{(x+1)(x-1)}=\frac{3ax(x-1)}{x^2-1};$$

$$\frac{4ax}{x^2-1}=\frac{4ax}{x^2-1}.$$

9. $a,\ \frac{a}{c},\ \frac{a-b}{a-c};\quad a=\frac{ac(a-c)}{c(a-c)};\quad \frac{a}{c}=\frac{a(a-c)}{c(a-c)};\quad \frac{a-b}{a-c}=\frac{c(a-b)}{c(a-c)}.$

10. $\frac{a-c}{(a+c)^2},\quad \frac{a+c}{(a-c)^2};$

$$\frac{a-c}{(a+c)^2}=\frac{(a-c)(a-c)^2}{(a+c)^2(a-c)^2}=\frac{(a-c)^3}{(a^2-c^2)^2};$$

$$\frac{a+c}{(a-c)^2}=\frac{(a+c)(a+c)^2}{(a-c)^2(a+c)^2}=\frac{(a+c)^3}{(a^2-c^2)^2}.$$

11. $\frac{a+c}{a-c},\quad \frac{a-c}{a+c},\quad \frac{a^2+c^2}{a^2-c^2};$ L. C. M. $=a^2-c^2$.

$(a^2-c^2)\div(a-c)=a+c;\quad (a^2-c^2)\div(a+c)=a-c.$

$$\frac{a+c}{a-c}=\frac{(a+c)(a+c)}{(a-c)(a+c)}=\frac{(a+c)^2}{a^2-c^2};$$

$$\frac{a-c}{a+c}=\frac{(a-c)(a-c)}{(a+c)(a-c)}=\frac{(a-c)^2}{a^2-c^2};$$

$$\frac{a^2+c^2}{a^2-c^2}\qquad\qquad=\frac{a^2+c^2}{a^2-c^2}.$$

12. $\frac{a}{a^2+1},\quad \frac{a^2}{a^2-1},\quad \frac{a^4}{a^4-1};$ L. C. M. $=a^4-1$.

$(a^4-1)\div(a^2+1)=a^2-1;\quad (a^4-1)\div(a^2-1)=a^2+1.$

$$\frac{a}{a^2+1}=\frac{a(a^2-1)}{(a^2+1)(a^2-1)}=\frac{a(a^2-1)}{a^4-1};$$

$$\frac{a^2}{a^2-1}=\frac{a^2(a^2+1)}{(a^2-1)(a^2+1)}=\frac{a^2(a^2+1)}{a^4-1};$$

$$\frac{a^4}{a^4-1}\qquad\qquad=\frac{a^4}{a^4-1}.$$

13. If we change the signs of the numerator and the second factor of the denominator of the second fraction, the value remains unchanged, and it assumes the form $\frac{-b}{(a-b)(b-c)}$.

NOTE.—The signs of the first fraction might have been changed, when the denominator of the second would become the common denominator.

ADDITION OF FRACTIONS.

Art. 139. (page 83.)

7. $\frac{2a}{3}+a-\frac{2x^2}{3ac}=\frac{2a^2c}{3ac}+\frac{3a^2c}{3ac}-\frac{2x^2}{3ac}=\frac{5a^2c-2x^2}{3ac}.$

9. $\frac{1}{m+n}+\frac{1}{m-n}=\frac{m-n}{m^2-n^2}+\frac{m+n}{m^2-n^2}=\frac{2m}{m^2-n^2}.$

10. $\frac{a}{a+b}+\frac{b}{a-b}=\frac{a^2-ab}{a^2-b^2}+\frac{ab+b^2}{a^2-b^2}=\frac{a^2+b^2}{a^2-b^2}.$

11. $\frac{a}{2a-2b}+\frac{b}{2b-2a}=\frac{a}{2a-2b}+\frac{-b}{2a-2b}=\frac{a-b}{2a-2b}=\frac{1}{2}.$

12. $\frac{a}{2c}+\frac{a-c}{ac}+\frac{c-a}{ac}=\frac{a}{2c}+\frac{a-c+c-a}{ac}=\frac{a}{2c}.$

13. $\frac{a^n}{2x}+\frac{x-a^{2n}}{a^nx}+\frac{a^{2n}x-2z^2}{2a^nx^2}=\frac{a^{2n}x}{2a^nx^2}+\frac{2x^2-2a^{2n}x}{2a^nx^2}+\frac{a^{2n}x-2z^2}{2a^nx^2}=\frac{2x^2-2z^2}{2a^nx^2}$
$=\frac{x^2-z^2}{a^nx^2}.$

14. $\frac{1+a}{1-a}+\frac{1-a}{1+a}=\frac{1+2a+a^2}{1-a^2}+\frac{1-2a+a^2}{1-a^2}=\frac{2(1+a^2)}{1-a^2}.$

15. $\frac{x-y}{xy}+\frac{y-z}{yz}+\frac{z-x}{xz}=\frac{xz-yz}{xyz}+\frac{xy-xz}{xyz}+\frac{yz-xy}{xyz}=0.$

16. $3a^2+\frac{x-3}{3}+4a^2+\frac{2a-z}{2a}=7a^2+\frac{2ax-6a}{6a}+\frac{6a-3z}{6a}=7a^2+\frac{2ax-3z}{6a}.$

17. $\frac{2}{z+1}+\frac{1+z^2}{z^2+z}=\frac{2z+1+z^2}{z^2+z}=\frac{(z+1)^2}{z(z+1)}=\frac{z+1}{z}.$

18. $\frac{1+x^2}{1-x^2}+\frac{1-x^2}{1+x^2}=\frac{1+2x^2+x^4}{1-x^4}+\frac{1-2x^2+x^4}{1-x^4}=\frac{2(1+x^4)}{1-x^4}.$

19. $\frac{a}{(a-b)(b-c)}+\frac{b}{(a-b)(c-b)}=\frac{a}{(a-b)(b-c)}\quad\frac{-b}{(a-b)(b-c)}$
$=\frac{a-b}{(a-b)(b-c)}=\frac{1}{b-c}.$

NOTE.—We change the signs of the numerator and second factor of the denominator of the second fraction, which does not alter either the value or sign of the fraction, but reduces the fractions to a common denominator.

20. $$\frac{(a+b)}{(b-c)(c-a)}+\frac{b+c}{(c-a)(a-b)}+\frac{c+a}{(a-b)(b-c)}=\frac{(a+b)(a-b)}{(b-c)(c-a)(a-b)}$$
$$+\frac{(b+c)(b-c)}{(b-c)(c-a)(a-b)}+\frac{(c+a)(c-a)}{(a-b)(b-c)(c-a)}=\frac{a^2-b^2}{(b-c)(c-a)(a-b)}$$
$$+\frac{b^2-c^2}{(b-c)(c-a)(a-b)}+\frac{c^2-a^2}{(b-c)(c-a)(a-b)}=\frac{a^2-b^2+b^2-c^2+c^2-a^2}{(a-b)(b-c)(c-a)}=0.$$

SUBTRACTION OF FRACTIONS.

Art. 140. (page 85.)

9. $$4c-\frac{a}{a-3}-\left(2c-\frac{a+3}{a}\right)=2c-\frac{a^2-a^2+9}{a(a-3)}=2c-\frac{9}{a(a-3)}.$$

10. $$\frac{a+b}{a}-\frac{b-a}{b}=\frac{ab+b^2-ab+a^2}{ab}=\frac{b^2+a^2}{ab}.$$

11. $$\frac{a}{a-b}-\frac{b}{a+b}=\frac{a^2+ab-ab+b^2}{a^2-b^2}=\frac{a^2+b^2}{a^2-b^2}.$$

12. $$\frac{1}{a-b}-\frac{b}{a^2-b^2}=\frac{a+b-b}{a^2-b^2}=\frac{a}{a^2-b^2}.$$

13 $$\frac{1}{1-a}-\frac{1}{1+a}=\frac{1+a-1+a}{1-a^2}=\frac{2a}{1-a^2}.$$

14. $$\frac{x+1}{x-1}-\frac{x-1}{x+1}=\frac{x^2+2x+1-x^2+2x-1}{x^2-1}=\frac{4x}{x^2-1}.$$

15. $$\frac{1+z^2}{1-z^2}-\frac{1-z^2}{1+z^2}=\frac{1+2z^2+z^4-1+2z^2-z^4}{1-z^4}=\frac{4z^2}{1-z^4}.$$

16. $$\frac{a}{a-x}+\frac{3a}{a+x}-\frac{2ax}{a^2-x^2}=\frac{a^2+ax+3a^2-3ax-2ax}{a^2-x^2}=\frac{4a^2-4ax}{a^2-x^2}$$
$$=\frac{4a(a-x)}{a^2-x^2}=\frac{4a}{a+x}.$$

17. $$\frac{1}{a+b}+\frac{b}{a^2-b^2}-\frac{a}{a^2+b^2}=\frac{a^3-a^2b+ab^2-b^3}{a^4-b^4}+\frac{a^2b+b^3}{a^4-b^4}-\frac{a^3-ab^2}{a^4-b^4}$$
$$=\frac{2ab^2}{a^4-b^4}.$$

18. $\dfrac{3}{1-2x}-\dfrac{7}{1+2x}+\dfrac{4-20x}{1-4x^2}=\dfrac{3+6x-7+14x+4-20x}{1-4x^2}=0.$

19. $\dfrac{a}{(a+b)(b-c)}-\dfrac{b}{(a+b)(c-b)}=\dfrac{a}{(a+b)(b-c)}-\dfrac{-b}{(a+b)(b-c)}$

$=\dfrac{a+b}{(a+b)(b-c)}=\dfrac{1}{b-c}.$

20. $\left(\dfrac{1}{m}+\dfrac{1}{n}\right)(a+b)-\left(\dfrac{a+b}{m}-\dfrac{a-b}{n}\right)=\dfrac{a+b}{m}+\dfrac{a+b}{n}-\dfrac{a+b}{m}+\dfrac{a-b}{n}$

$=\dfrac{2a}{n}.$

MULTIPLICATION OF FRACTIONS.

Art. 142. (page 87.)

9. $\dfrac{x+y}{x^2-2xy+y^2}\times x^2-y^2=\dfrac{x+y}{(x-y)^2}\times(x+y)(x-y)=\dfrac{(x+y)^2}{x-y}.$

10. $\dfrac{5a^2x}{a-1}\times a^2-1=\dfrac{5a^2x}{a-1}\times(a+1)(a-1)=5a^3x+5a^2x.$

11. $\dfrac{3a^2z^3}{x^3-x}\times 2a(x-1)=\dfrac{3a^2z^3}{x(x+1)(x-1)}\times 2a(x-1)=\dfrac{6a^3z^3}{x(x+1)}.$

Art. 143. (page 88.)

10. $\dfrac{1-a^2}{6a^3}\times\dfrac{4ab^3}{1-a}=\dfrac{(1+a)(1-a)}{6a^3}\times\dfrac{4ab^3}{1-a}=\dfrac{2b^3(1+a)}{3a^2}.$

11. $\dfrac{(a-b)^2}{a+b}\times\dfrac{(a+b)^2}{a-b}=(a+b)(a-b)=a^2-b^2.$

12. $\left(a-\dfrac{a}{c}\right)\times\dfrac{2bc}{3a}=\dfrac{ac-a}{c}\times\dfrac{2bc}{3a}=\dfrac{2b(c-1)}{3}.$

13. $\left(a+\dfrac{a}{x}\right)\times\left(a-\dfrac{a}{x}\right)=a^2-\dfrac{a^2}{x^2}$, by Theorem III.

14. $\dfrac{n^2-z^2}{3m^2}\times\dfrac{6n^2}{n+z}=\dfrac{(n+z)(n-z)6n^2}{3m^2(n+z)}=\dfrac{2n^2(n-z)}{m^2}.$

15. $\frac{a^2+ab}{(1+b)^2}\times\frac{c+bc}{a+b}=\frac{a(a+b)(1+b)c}{(1+b)^2(a+b)}=\frac{ac}{1+b}.$

16. $\frac{ac+bc}{(a-b)^2}\times\frac{a^2-ab}{c^2}=\frac{c(a+b)(a-b)a}{(a-b)^2c^2}=\frac{a(a+b)}{c(a-b)}.$

17. Multiply the two binomials without reducing to single terms.

18. $\frac{n}{m+n}\times\frac{m^2-n^2}{m^2}\times\frac{m}{m-n}=\frac{mn(m^2-n^2)}{m^2(m^2-n^2)}=\frac{n}{m}.$

19. $\frac{1-x^2}{1-c}\times\frac{1-c^2}{x+x^2}=\frac{(1-x)(1+x)(1-c)(1+c)}{(1-c)(1+x)x}=\frac{(1-x)(1+c)}{x}.$

20. $\frac{a(a-b)}{a^2+2ab+b^2}\times\frac{a(a+b)}{a^2-2ab+b^2}=\frac{a^2}{(a+b)(a-b)}=\frac{a^2}{a^2-b^2}.$

21. $\frac{a^4-b^4}{a^2-2ab+b^2}\times\frac{a-b}{a^2+ab}=\frac{(a+b)(a-b)(a^2+b^2)(a-b)}{(a-b)(a-b)(a+b)a}=\frac{a^2+b^2}{a}.$

22. $\left(\frac{a^2}{b-c}-\frac{b^2}{b-c}\right)\times\left(\frac{b}{a^2-b^2}-\frac{c}{a^2-b^2}\right)=\frac{a^2-b^2}{b-c}\times\frac{b-c}{a^2-b^2}=1.$

23. $\left(\frac{a+b}{b-c}\right)(a-b)\left(\frac{b-c}{a+b}\right)\left(\frac{1}{a-b}\right)=\frac{(a+b)(a-b)(b-c)}{(b-c)(a+b)(a-b)}=1.$

DIVISION OF FRACTIONS.

Art. 145. (page 90.)

8. $\frac{x^2-1}{ab^2}\div a(x+1)=\frac{(x+1)(x-1)}{ab^2}\div a(x+1)=\frac{x-1}{a^2b^2}.$

9. $\frac{a^3-ab^2}{a-c}\div c^2(a-b)=\frac{a(a+b)(a-b)}{a-c}\div c^2(a-b)=\frac{a(a+b)}{c^2(a-c)}.$

10. $\frac{a^3x-ax^3}{c^n}\div ac^n+c^nx=\frac{ax(a+x)(a-x)}{c^n}\div(a+x)c^n=\frac{ax(a-x)}{c^{2n}}.$

Art. 146. (page 91.)

10. $\frac{a+1}{2a}\div\frac{a-1}{4a^2}=\frac{a+1}{2a}\times\frac{4a^2}{a-1}=\frac{2a(a+1)}{a-1}.$

11. $\frac{a^2-x^2}{a-1} \div \frac{a+x}{a(a-1)} = \frac{a^2-x^2}{a-1} \times \frac{a(a-1)}{a+x} = a(a-x)$

12. $\frac{ax^2-bx^3}{3a} \div \frac{5cx^2}{6ab} = \frac{ax^2-bx^3}{3a} \times \frac{6ab}{5cx^2} = \frac{2b(a-bx)}{5c}.$

13. $\frac{4a^3n}{a^2-b^2} \div \frac{an^3}{a+b} = \frac{4a^3n}{a^2-b^2} \times \frac{a+b}{an^3} = \frac{4a^2}{n^2(a-b)}.$

14. $\left(1+\frac{1}{x}\right) \div 1 - \frac{1}{x^2} = \frac{x+1}{x} \times \frac{x^2}{x^2-1} = \frac{x}{x-1}.$

15. $\left(1+\frac{a^n}{x^n}\right) \div \left(1+\frac{x^n}{a^n}\right) = \frac{x^n+a^n}{x^n} \times \frac{a^n}{a^n+x^n} = \frac{a^n}{x^n}.$

16. $\frac{(x-1)^2}{a^2-1} \div \frac{x^2+x-2}{a-1} = \frac{(x-1)^2}{(a+1)(a-1)} \times \frac{a-1}{(x+2)(x-1)} = \frac{x-1}{(a+1)(x+2)}.$

17. $\frac{a^2-b^2}{a^2+ax} \div \frac{(a-b)^2}{a+x} = \frac{a^2-b^2}{a(a+x)} \times \frac{a+x}{(a-b)^2} = \frac{a+b}{a(a-b)}.$

18. $\frac{x^2+3x+2}{x+3} \div \frac{x^2+x}{x+3} = \frac{(x+2)(x+1)}{x+3} \times \frac{x+3}{x(x+1)} = \frac{x+2}{x} = 1+\frac{2}{x}.$

19. $\frac{x^2-5x+6}{x+4} \div \frac{x-2}{x^2+x-12} = \frac{(x-3)(x-2)}{x+4} \times \frac{(x+4)(x-3)}{x-2} = (x-3)^2.$

20. $1-\frac{a^{2n}}{x^{2n}} \div 1+\frac{a^n}{x^n} = 1-\frac{a^n}{x^n}$, by Theorem III.

COMPLEX FRACTIONS.

Art. 147. (page 93.)

3. $\dfrac{\frac{2a^2}{c^3}}{\frac{4ax}{bc}} = \frac{2a^2}{c^3} \times \frac{bc}{4ax} = \frac{ab}{2c^2x}.$

4. $\dfrac{\frac{a}{2}-\frac{b}{a}}{1+\frac{1}{a}} = \left(\frac{a}{2}-\frac{b}{a}\right) \div \left(1+\frac{1}{a}\right) = \frac{a^2-2b}{2a} \times \frac{a}{a+1} = \frac{a^2-2b}{2(a+1)}.$

5. $$\frac{1+\frac{1}{c}}{a+\frac{1}{a}}=\left(1+\frac{1}{c}\right)\div\left(a+\frac{1}{a}\right)=\frac{c+1}{c}\times\frac{a}{a^2+1}=\frac{a(c+1)}{c(a^2+1)}.$$

6. $$\frac{\frac{c}{c-1}-1}{1-\frac{c}{c+1}}=\left(\frac{c}{c-1}-1\right)\div\left(1-\frac{c}{c+1}\right)=\frac{1}{c-1}\times\frac{c+1}{1}=\frac{c+1}{c-1}.$$

7. $$\frac{\frac{a+b}{x+y}}{\frac{a^2-b^2}{x^2-y^2}}=\frac{a+b}{x+y}\times\frac{x^2-y^2}{a^2-b^2}=\frac{x-y}{a-b}.$$

8. $$1-\frac{1}{1+\frac{1}{a}}=1-\frac{a}{a+1}=\frac{a+1-a}{a+1}=\frac{1}{a+1}.$$

9. $$\frac{1}{1-\frac{1}{1+\frac{1}{n}}}=\frac{1}{1-\frac{n}{n+1}}=\frac{n+1}{n+1-n}=n+1.$$

10. $$\frac{a-1+\frac{6}{a-6}}{a-2+\frac{3}{a-6}}=\frac{a^2-7a+12}{a^2-8a+15}=\frac{(a-3)(a-4)}{(a-3)(a-5)}=\frac{a-4}{a-5}.$$

VANISHING FRACTIONS.

Art. 148. (page 94.)

2. $\dfrac{x^2-1}{x-1}=x+1.$ If $x=1$ $x+1=2.$

3. $\dfrac{x^3-1}{x-1}=x^2+x+1=1+1+1=3.$

4. $\dfrac{x^3-a^3}{x-a}=x^2+ax+a^2=a^2+a^2+a^2=3a^2.$

5. $\dfrac{x^3-a^3}{x^2-a^2}=\dfrac{x^2+ax+a^2}{x+a}=\dfrac{a^2+a^2+a^2}{a+a}=\dfrac{3a^2}{2a}=\dfrac{3a}{2}.$

6. $\dfrac{x^4-a^4}{x-a}=x^3+ax^2+a^2x+a^3=4a^3.$

7. $\dfrac{(x-a)^2}{x^3-a^3}=\dfrac{x-a}{x^2+ax+a^2}=\dfrac{0}{3a^2}=0.$

8. $\dfrac{x-x^6}{1-x}=x+x^2+x^3+x^4+x^5=1+1+1+1+1=5.$

9. $\dfrac{x^2+2x-15}{x^2+4x-21}=\dfrac{x+5}{x+7}=\dfrac{3+5}{3+7}=\dfrac{8}{10}=\dfrac{4}{5}.$

10. $\dfrac{1-x^m}{1-x}=1+x+x^2 \ldots x^{m-1}=m.$

Since there are evidently $m-1$ powers of x in the quotient and also an absolute term, there will be m terms in the expression, and each power of x being 1, there will be m *ones*, which equals m.

11. $\dfrac{x^n-a^n}{x-a}=x^{n-1}+ax^{n-2}+\ \ldots\ a^{n-2}x+a^{n-1}=a^{n-1}+a^{n-1}\ \ldots$ $a^{n-1}+a^{n-1}=na^{n-1}$, there being n terms, as was shown in the last example.

SIMPLE EQUATIONS.

NOTE.—In clearing an equation of fractions, it is often best to indicate some of the operations, as may be seen in the following example:

$$\frac{a}{3}-\frac{6-x}{4}=\frac{x-2}{3}.$$

Multiplying both numbers by 12, the least common multiple of the denominators (Art. 162, Case I.), we have,

$$4a-3(6-x)=4(x-2);$$

or, $\quad 4a-18+6x=4x-8.$

Art. 162. CASE II. (page 98.)

7. Given, $\quad \dfrac{x}{a-b}-b=\dfrac{c}{2},$

Multiplying by $2(a-b)$, $\quad 2x-2b(a-b)=c(a-b),$

Expanding, $\quad 2x-2ab+2b^2=ac-bc.$

9. Given, $$\frac{2x}{a+b}-\frac{3x}{a-b}=4;$$

Multiplying by $(a+b)(a-b)$, $$2x(a-b)-3x(a+b)=4(a+b)(a-b);$$

Expanding, $$2ax-2bx-3ax-3bx=4a^2-4b^2.$$

10. Given, $$\frac{3x-4}{a}=2-3a^{-1};$$

Multiplying by a, $$3x-4=2a-3a^0;$$

whence (Art. 93), $$3x-4=2a-3.$$

11. Given, $$\frac{x+1}{x-1}-\frac{x-1}{x+1}=\frac{3a}{x^2-1};$$

Multiplying by (x^2-1), $$(x+1)(x+1)-(x-1)(x-1)=3a;$$

or, $$(x+1)^2-(x-1)^2=3a.$$

12. Given, $$\frac{2-\frac{x}{2}}{\frac{x}{3}}=\frac{1-\frac{1}{2}}{\frac{2}{3}};$$

Multiplying by $\frac{2x}{3}$, $$2\left(2-\frac{x}{2}\right)=x(1-\tfrac{1}{2});$$

Expanding, $$4-x=x-\frac{x}{2};$$

and multiplying by 2, $$8-2x=2x-x.$$

13. Given, $$\frac{x+3}{x-3}-\frac{x-3}{x+3}=6\tfrac{6}{7};$$

Multiplying by $7(x^2-9)$, $$7(x+3)(x+3)-7(x-3)(x-3)=48(x^2-9)$$

whence, $$7(x+3)^2-7(x-3)^2=48(x^2-9).$$

Art. 167. CASE I. (page 101.)

14. Given, $$\frac{2x}{7}-\frac{3x}{4}=\frac{7x}{6}-11\tfrac{5}{12};$$

Multiplying by 84, $$24x-63x=98x-959;$$
Transposing, $$24x-63x-98x=-959;$$
Uniting terms, $$-137x=-959;$$
and dividing by -137, $$x=7.$$

16. Given, $$\frac{2x}{3}+\frac{x-1}{6}=\frac{3x+1}{2}-10;$$

Clearing of fractions, $$4x+x-1=9x+3-60;$$
Transposing, $$4x+x-9x=1+3-60;$$
whence, $$x=14.$$

17. Given, $\frac{x+3}{2}+\frac{x}{3}=4-\frac{x-5}{4}$;

Clearing of fractions, $6(x+3)+4x=48-3(x-5)$;
Expanding, $6x+18+4x=48-3x+15$;
Transposing, $6x+4x+3x=48+15-18$;
whence, $x=3\frac{6}{13}$.

18. Given, $x+\frac{2x-4}{3}=12-\frac{3x-5}{2}$;

Clearing of fractions, $6x+4x-8=72-9x+15$;
Transposing, $6x+4x+9x=72+15+8$;
whence, $x=5$.

19. Given, $\frac{x-5}{3}+\frac{x}{2}=12-\frac{x-10}{3}$;

Clearing of fractions, $2x-10+3x=72-2x+20$;
Transposing, $2x+3x+2x=72+20+10$;
whence, $x=14\frac{4}{7}$.

29. Given, $\frac{x+3}{2}-\frac{x-2}{3}=\frac{3x-5}{12}+\frac{1}{4}$;

Clearing of fractions, $6x+18-4x+8=3x-5+3$;
Transposing, $6x-4x-3x=-5+3-18-8$;
Uniting terms, $-1x=-28$;
Dividing by -1, $x=28$.

Art. 167. CASE II. (page 102.)

7. Given, $ax+b=\frac{x}{a}+\frac{1}{b}$;

Multiplying by ab, $a^2bx+ab^2=bx+a$;
Transposing, $a^2bx-bx=a-ab^2$;
Factoring, $b(a^2-1)x=a(1-b^2)$;
whence, $x=\frac{a(1-b^2)}{b(a^2-1)}$.

8. Given, $\frac{x}{a}+\frac{x}{b-a}=\frac{a}{b+a}$;

Multiplying by $a^2(b^2-a^2)$, $(b^2-a^2)x+(b+a)ax=a^2(b-a)$;
Expanding, $b^2x-a^2x+abx+a^2x=a^2(b-a)$;
Uniting terms and factoring, $b(b+a)x=a^2(b-a)$;
whence, $x=\frac{a^2(b-a)}{b(b+a)}$.

9. Given, $\frac{1-x}{1+x}=1-\frac{1}{c}$;

Multiplying by $c(1+x)$, $c(1-x)=c(1+x)-(1+x)$;

Expanding, $c-cx=c+cx-1-x$;

Transposing, $x-2cx=c-c-1=-1$;

Multiplying by -1, $2cx-x=1$;

whence, $x=\frac{1}{2c-1}$.

10. Given, $x+a=\frac{x^2}{a+x}$;

Multiplying by $(x+a)$, $x^2+2ax+a^2=x^2$;

Transposing, $x^2-x^2+2ax=-a^2$;

Uniting terms, $2ax=-a^2$;

and dividing by $2a$, $x=-\frac{a^2}{2a}=-\frac{a}{2}$.

11. Given, $(a+x)(b+x)-a(b+c)=\frac{a^2c}{b}+x^2$;

Expanding, $ab+bx+ax+x^2-ab-ac=\frac{a^2c}{b}+x^2$;

Transposing, $bx+ax+x^2-x^2=\frac{a^2c}{b}-ab+ab+ac$;

Uniting terms, $(a+b)x=\frac{a^2c}{b}+ac=\frac{ac(a+b)}{b}$;

and dividing by $a+b$, $x=\frac{ac}{b}$.

12. Given, $\frac{a+b}{a-x}+\frac{b+c}{c-x}=\frac{a-b}{a-x}$;

Multiplying by $(a-x)(c-x)$, $(a+b)(c-x)+(b+c)(a-x)$
$=(a-b)(c-x)$;

Expanding, $ac+bc-ax-bx+ab+ac-bx-cx=ac-bc$
$-ax+bx$;

Transposing, $-ax-bx-bx-cx+ax-bx=ac-bc-ac-bc$
$-ab-ac$;

Uniting terms, $-cx-3bx=-ab-2bc-ac$;

Multiplying by -1, $(c+3b)x=ab+2bc+ac$;

whence, $x=\frac{ab+2bc+ac}{c+3b}$.

Art. 169. (page 103.)

6. Given, $\frac{2x}{3}-3\frac{1}{3}+13=13\frac{11}{12}-\frac{3x}{4}$;

Transposing and uniting, $\frac{2x}{3}=4\frac{1}{4}-\frac{3x}{4}$;

Clearing of fractions, $8x=51-9x$;

whence, $x=3$.

7. Given, $\frac{3x}{4}-4\frac{1}{2}+a=a+\frac{x}{5}+1$;

Transposing and uniting, $\frac{3x}{4}=\frac{x}{5}+5\frac{1}{2}$;

Clearing of fractions, $15x=4x+110$;

whence, $x=10$.

8. Given, $2ax+3m-\frac{1}{3}cx=ax+2m+\frac{2}{3}cx+n$;

Transposing and uniting, $ax-cx=n-m$;

whence, $x=\frac{n-m}{a-c}$.

9. Given, $3ax-2bx-\frac{1}{3}c-\frac{1}{4}mx=\frac{2}{3}c+\frac{3}{4}mx-n-bx+2ax$;

Transposing and uniting, $ax-bx-mx=c-n$;

whence, $x=\frac{c-n}{a-b-m}$.

Art. 170. (page 104.)

4. Given, $\frac{1}{3}x+\frac{1}{5}x+\frac{1}{6}x=42$;

Clearing of fractions, $10x+6x+5x=42\times 30$;

Uniting terms, $21x=42\times 30$;

Dividing by 21, $x=2\times 30=60$.

5. Given, $3x+\frac{2x}{3}+\frac{5x}{6}=54$;

Clearing of fractions, $18x+4x+5x=54\times 6$;

Uniting terms, $27x=54\times 6$;

whence, $x=2\times 6=12$.

Art. 171. (page 105.)

4. Given, $$\frac{x-5}{3}+\frac{2(x-5)}{4}=\frac{3}{5}(x-5)+14.$$

Let $$y=x-5;$$

Substituting, $$\frac{y}{3}+\frac{2y}{4}=\frac{3}{5}y+14;$$

Clearing of fractions, $$20y+30y=36y+840;$$

whence, $$y=60;$$

or, $$x-5=60;$$

and $$x=65.$$

5. Given, $$\frac{2x+4}{3}-\frac{x-3}{4}=\frac{x+2}{3}+3\tfrac{1}{3}.$$

Let $$y=x+2;$$

then, $$y-5=x-3;$$

Substituting, $$\frac{2y}{3}-\frac{y-5}{4}=\frac{y}{3}+3\tfrac{1}{3};$$

Clearing of fractions, $$8y-3y+15=4y+40;$$

Transposing, $$y=25;$$

or, $$x+2=25;$$

whence, $$x=23.$$

6. Given, $$\frac{x+c}{3}-\frac{3(x+c)}{4}=\frac{1}{3}(x+c)-c.$$

Let y represent $x+c$;

then, $$\frac{y}{3}-\frac{3y}{4}=\frac{1}{3}y-c;$$

Clearing of fractions, $$4y-9y=4y-12c;$$

whence, $$y=\tfrac{4}{3}c;$$

or, $$x+c=\tfrac{4}{3}c;$$

whence, $$x=\tfrac{1}{3}c=\frac{c}{3}.$$

NOTE.—Many persons prefer to use the binomial instead of substituting for it a single letter.

Art. 172. (page 105.)

2. Given, $\frac{7-9x}{12}-\frac{12-4x}{5-3x}=\frac{15-6x}{8}-\frac{7}{24}$;

Separating the terms, $\frac{7}{12}-\frac{9x}{12}-\frac{12-4x}{5-3x}=\frac{15}{8}-\frac{6x}{8}-\frac{7}{24}$;

Transposing and reducing, $-1=\frac{12-4x}{5-3x}$;

Clearing of fractions, $-5+3x=12-4x$;

whence, $x=2\frac{3}{7}$.

3. Given, $\frac{6x-15}{9}-\frac{10x-17}{15}=\frac{4x-15}{3-2x}+\frac{2}{15}$;

Separating the terms, $\frac{6x}{9}-\frac{15}{9}-\frac{10x}{15}+\frac{17}{15}=\frac{4x-15}{3-2x}+\frac{2}{15}$;

Transposing and reducing, $-\frac{2}{3}=\frac{4x-15}{3-2x}$;

Clearing of fractions, $-6+4x=12x-45$;

whence, $x=4\frac{7}{8}$.

4. Given, $\frac{x-16}{18}-\frac{17-4x}{9}=\frac{5x}{7}-\frac{4-26x}{32-17x}-\frac{3x}{14}$;

Separating the terms, $\frac{x}{18}-\frac{16}{18}-\frac{17}{9}+\frac{4x}{9}=\frac{5x}{7}-\frac{4-26x}{32-17x}-\frac{3x}{14}$;

Reducing, $-\frac{25}{9}=-\frac{4-26x}{32-17x}$;

Transposing, $\frac{4-26x}{32-17x}=\frac{25}{9}$;

Clearing of fractions, $36-234x=800-425x$;

whence, $x=4$.

PROBLEMS IN SIMPLE EQUATIONS.

Art. 176. CASE I. (page 107.)

3. Let $x=$ length of shortest piece;

then, $5x=$ length of longest piece;

and $2x=$ length of other piece.

Then, $x+5x+2x=96$;

whence, $x=12$, length of shortest;

and $2x=24$, length of second;

and $5x=60$, length of longest.

4. Let $x=$ first part;
then, $mx=$ second part;
and $mnx=$ third part.

Then, $x+mx+mnx=a$;

whence, $x=\dfrac{a}{1+m+mn}$;

and $mx=\dfrac{ma}{1+m+mn}$;

and $mnx=\dfrac{mna}{1+m+mn}$.

CASE II. (page 107.)

3. Let $x=$ Hinkley's share;

then, $\dfrac{2x}{3}=$ Harry's share;

and $\frac{3}{4}$ of $\dfrac{2x}{3}=\dfrac{x}{2}=$ Harvey's share.

Then, $x+\frac{2}{3}x+\frac{1}{2}x=2782$;
Clearing of fractions, $6x+4x+3x=16692$;
whence, $x=1284$, Hinkley's;
and $\frac{2}{3}x=856$, Harry's;
and $\frac{1}{2}x=642$, Harvey's.

4. Let $x=$ first part;

and $\dfrac{nx}{m}=$ second part.

Then, $x+\dfrac{nx}{m}=a$;

Clearing of fractions, $mx+nx=am$;

whence, $x=\dfrac{am}{m+n}$, first part;

and $\dfrac{nx}{m}=\dfrac{an}{m+n}$, second part.

CASE III. (page 108.)

3. Let $x=$ third number;

then, $\dfrac{2(x-20)}{3}=$ second number;

and $\dfrac{4(x-20)}{3}+15=$ third number.

Then, $x+\frac{2(x-20)}{3}+\frac{4(x-20)}{3}+15=215$;

Uniting, $x+2(x-20)=200$;

Expanding, $x+2x-40=200$;

whence, $x=80$, third number;

and $\frac{2(x-20)}{3}=40$, second number;

and $\frac{4(x-20)}{3}+15=95$, first number.

4. Let $x=$ first part;

and $mx+n=$ second part.

Then, $x+mx+n=a$;

Transposing, $x+mx=a-n$.

whence, $x=\frac{a-n}{1+m}$, first part;

and $mx+n=\frac{ma+n}{1+m}$, second part.

CASE IV. (page 109.)

4. Let $x=$ the number of sheep A bought;

and $2x=$ the number of lambs B bought;

then, $12x=$ cost of sheep;

and $16x=$ cost of lambs.

Then, $12x+40=16x-40$;

Transposing, $4x=80$;

whence, $12x=240$, what A spent;

and $16x=320$, what B spent.

5. Let $x=$ number of beggars;

then, $mx=$ what he gave them;

and $nx=$ what he would have given them.

Therefore, $mx+a=nx+b$;

Transposing, $mx-nx=b-a$;

whence, $x=\frac{b-a}{m-n}$, number of beggars;

$$mx+a=\frac{mb-ma}{m-n}+a=\frac{bm-an}{m-n};$$

CASE V. (page 109.)

4. Let $x=$ the time;

$2x=$ what A does; $3x=$ what B does; $4x=$ what C does.

Therefore, $2x+3x+4x=1$;

and $x=\frac{1}{9}$ of a day.

5. Let x = the time;

$\frac{x}{a}$ = what A does; $\frac{x}{b}$ = what B does; $\frac{x}{c}$ = what C does.

Therefore, $$\frac{x}{a}+\frac{x}{b}+\frac{x}{c}=1;$$

Clearing of fractions, $$bcx+acx+abx=abc;$$

whence, $$x=\frac{abc}{bc+ac+ab}.$$

CASE VI. (page 110.)

2. Let x = the number broken;

and $144-x$ = the number left;

then, $2\frac{1}{2}x$ = sum forfeited;

and $\frac{144}{4}$ = the sum earned.

Then, $36-2\frac{1}{2}x=26$;

Transposing, $2\frac{1}{2}x=10$;

whence, $x=4$.

NOTE.—This example may be taken also to mean that she receives nothing for the broken eggs, but forfeits 10 cents. In that case she should receive 25 cents.

3. Let x = number of idle days;

and $40-x$ = the number of working days;

then, $250(40-x)$ = sum received;

and 40×50 = sum paid for board.

Then, $250(40-x)-2000=5000$.

Transposing, $250x=3000$;

whence, $x=12$, number of idle days.

4. Let x = the number of working days;

and $n-x$ = the number of idle days;

then, ax = what he earns;

and $b(n-x)$ = what he forfeits.

Then, $ax-b(n-x)=c$;

whence, $x=\frac{c+bn}{a+b}$, number of working days;

and $n-x=\frac{an-c}{a+b}$, number of idle days.

CASE VII. (page 111.)

2. Let $x=$ length of body;

$\frac{x}{4}+4=$ length of tail;

and $\frac{x}{8}+2+3=$ length of head.

Then, by the last condition, $x=2\left(\frac{x}{4}+4+\frac{x}{8}+2+3\right)$;

Reducing. $x=\frac{x}{2}+\frac{x}{4}+18$;

whence, $\frac{x}{4}=18$;

then, $x=72$, length of body;

and $\frac{x}{2}=36$, length of head and tail;

whence, $36+72=108$, length of whale.

3. Let $x=$ the number of infantry;

$\frac{x}{6}+370=$ the number of cavalry;

and $\frac{1}{3}\left(\frac{x}{6}+370\right)-60=$ the number of artillery.

Then, by the last condition, $x=4\left(\frac{x}{6}+370+\frac{1}{3}\left(\frac{x}{6}+370\right)-60\right)$;

Reducing, $x=4\left(\frac{4}{3}\left(\frac{x}{6}+370\right)-60\right)=\frac{8x}{9}+\frac{5920}{3}-240.$

or, $\frac{x}{9}=\frac{5200}{3}$;

Clearing of fractions, $x=15600$, infantry;

whence, $\frac{x}{6}+370=2970$, cavalry;

and $\frac{1}{3}\left(\frac{x}{6}+370\right)-60=930$, artillery;

then, $15600+2970+930=19500$, number in corps.

4. Let $x=$ length of body;

and $a+\frac{x}{n}=$ length of tail.

Then, by the last condition, $x=a+a+\frac{x}{n}$;

Clearing of fractions, $nx-x=2an$;

whence, $x=\frac{2an}{n-1}$, length of body, or head and tail;

and $2\frac{(2an)}{n-1}=\frac{4an}{n-1}$, length of fish.

CASE VIII. (page 111.)

3. Let x = number of miles he rides an hour;

then, $\dfrac{24}{x}$ = time in going;

and $\dfrac{24}{3} = 8$ = time in returning.

Then, $\dfrac{24}{x} + 8 = 11;$

Clearing of fractions, $24 = 3x;$

whence, $x = 8$, number of miles an hour.

4. Let x = distance he rides;

then, $\dfrac{x}{a}$ = time in going;

and $\dfrac{x}{c}$ = time in returning.

Then, $\dfrac{x}{a} + \dfrac{x}{c} = n;$

Clearing of fractions, $cx + ax = acn;$

whence, $x = \dfrac{acn}{a+c}.$

5. Let x = distance it goes;

then, $\dfrac{x}{a+c}$ = time in going;

and $\dfrac{x}{a-c}$ = time in returning.

Then, $\dfrac{x}{a+c} + \dfrac{x}{a-c} = n;$

Clearing of fractions, $2ax = (a^2 - c^2)n;$

whence, $x = \dfrac{(a^2 - c^2)}{2a}n.$

CASE IX. (page 112.)

3. Let x = number withdrawing;

then, $\frac{15}{15} = 1$ = share of each by first condition;

and $\dfrac{15}{15-x}$ = share of each by second condition.

Then, $\dfrac{15}{15-x} - 1 = \frac{1}{2};$

Clearing of fractions, $30 - 30 + 2x = 15 - x;$

whence, $x = 5$, number withdrawing.

4. Let x = sum to be paid;

then, $\frac{x}{n}$ = share of each by first condition;

and $\frac{x}{m+n}$ = share of each by second condition.

Then, $\frac{x}{n} - \frac{x}{m+n} = a$;

Clearing of fractions, $mx = an(m+n)$;

whence, $x = \frac{a(n^2+mn)}{m}$.

5. Let x = number who went;

and $n-x$ = number who remained;

then, $\frac{a}{n}$ = share of each by first condition;

and $\frac{a}{x}$ = share of each by second condition.

Then, $\frac{a}{x} - \frac{a}{n} = b$;

Clearing of fractions, $na - ax = bnx$;

whence, $x = \frac{an}{a+bn}$, the number who went;

and $n - x = \frac{bn^2}{a+bn}$, the number who remained.

CASE X. (page 113.)

2. Let x = his age.

Then, $x + \frac{2x}{5} + \frac{13}{5} = 4(x-13)$;

Clearing of fractions, $5x + 2x + 13 = 20x - 260$;

Collecting, $13x = 273$;

whence, $x = 21$, his age.

3. Let x = his original stock.

Then, $(x-1400) + \frac{4}{5}(x-1400) = \frac{3x}{4}$.

Collecting, $\frac{9}{5}(x-1400) = \frac{3x}{4}$.

Clearing of fractions, $36x - 50400 = 15x$;

whence, $x = 2400$.

4. Let $x =$ original sum;

then, $(x+a) - \frac{x+a}{n} = mx,$

Clearing of fractions, $nx + an - x - a = mnx;$

whence, $x = \frac{a(n-1)}{mn-n+1}.$

CASE XI. (page 113.)

3. Let $x =$ number of ounces of gold at first.

Then, $\frac{6}{x+56} = \frac{\frac{2}{5}}{10} = \frac{1}{25};$

Clearing of fractions, $150 = x + 56;$

whence, $x = 94.$

4. Let $x =$ number of ounces in the mixture.

Then, $\frac{4}{x+12} = \frac{\frac{2}{3}}{12} = \frac{1}{18};$

Clearing of fractions, $72 = x + 12;$

whence, $x = 60.$

5. Let $5x =$ the sum;

then, $3x =$ the gold;

and $2x =$ the silver.

Therefore, $3x + 24 = 6x;$

whence, $x = 8;$

and $5x = 40$, the sum.

6. Let $x =$ number of lbs. of salt to be added.

Then, $\frac{b+x}{a+x} = \frac{n}{m};$

Clearing of fractions, $mb + mx = an + nx;$

whence, $x = \frac{an - bm}{m-n}.$

CASE XII. (page 114.)

3. Let $x =$ number of days the second labors;

and $\frac{x}{3} + 6 =$ number of days the first labors;

also, $84 - 48 = 36 =$ money the second receives;

then, $\frac{36}{x} =$ what the second receives per day;

and $\frac{48}{\frac{x}{3}+6} =$ what the first man receives per day.

Therefore, $\dfrac{48}{\dfrac{x}{3}+6}=\dfrac{36}{x}$;

Clearing of fractions, $48x=12x+216$;
whence, $36x=216$;
or, $x=6$, number of days the second works.
and $\dfrac{x}{3}+6=8$, number of days the first works.

4. Let $x=$ the stock;
then, $\dfrac{x}{n}-b=$ A's stock;
and $\dfrac{(n-1)x}{n}+b=$ B's stock.
Then, $\dfrac{a}{x}=$ gain on \$1;
and $\dfrac{c}{\dfrac{x}{n}-b}=$ gain on \$1.

Therefore, $\dfrac{a}{x}=\dfrac{c}{\dfrac{x}{n}-b}$;
Clearing of fractions, $ax-abn=cnx$;
whence, $x=\dfrac{abn}{a-cn}$, stock;
and $\dfrac{x}{n}-b=\dfrac{bcn}{a-cn}$, A's share;
also, $\dfrac{abn}{a-cn}-\dfrac{bcn}{a-cn}=\dfrac{(a-c)bn}{a-cn}$, B's share.

CASE XIII. (page 115.)

2. Let $x=$ time past midnight;
and $\dfrac{3x}{7}=$ time past noon.

Then, $x-\dfrac{3x}{7}=12$;
whence, $x=21$ hours past midnight, or 9 P. M.

3. Let $x=$ time to noon;
and $\dfrac{3x}{5}=$ time past midnight.

Then, $x+\dfrac{3x}{5}=12$;
whence, $x=7\frac{1}{2}$ hours to noon, or $4\frac{1}{2}$ A. M.

4. Let x = time to midnight;

and $\frac{3x}{4}$ = time past 10 A. M.

Then, $x+\frac{3x}{4}=14;$

whence, $x=8$ hours to midnight, or 4 P. M.

5. Let x = time past 4 A. M.

and $\frac{4x}{5}$ = time to 10 P. M.

Then, $x+\frac{4x}{5}=18;$

whence, $x=10$ hours past 4 A. M., or 2 P. M.

6. Let x = time past midnight;

and $\frac{x}{n}$ = time to noon.

Then, $x+\frac{x}{n}=12;$

whence, $x=\frac{12n}{n+1}$ A. M.

CASE XIV. (page 115.)

2. Let x = distance the minute-hand goes;

then, $\frac{x}{12}$ = distance the hour-hand goes.

Then, $x-\frac{x}{12}=15$, number of minute-spaces they are apart at 3 o'clock;

whence, $x=16\frac{4}{11}$, $\therefore$ it is $16\frac{4}{11}$ minutes past 3.

3. Let x = distance the minute-hand goes;

then, $\frac{x}{12}$ = distance the hour-hand goes.

Now, at 4 o'clock the hands are 20 minute-spaces apart; hence to be 5 minute-spaces apart the minute-hand must gain $20-5$, or 15 minute-spaces.

Hence, $x-\frac{x}{12}=15;$

and, $x=16\frac{4}{11}$, $\therefore$ it is $16\frac{4}{11}$ minutes past 4.

NOTE.—They will also be 5 minute-spaces apart after they are together, in which case the minute-hand gains 25 minutes, and the time is $27\frac{3}{11}$ minutes past 4.

5

4. Let $x =$ distance the minute-hand goes;

then, $\frac{x}{12} =$ distance the hour-hand goes.

Then, $x - \frac{x}{12} = 20$, number of minute-spaces they are apart at 4 o'clock;

whence, $x = 21\frac{9}{11}$ minute-spaces.

Since the hands are together at $21\frac{9}{11}$ minutes past 4, they will be 5 minutes of time apart 5 minutes before they are together, and also 5 minutes after they are together; therefore they will be 5 minutes apart, $21\frac{9}{11} - 5$, or $16\frac{9}{11}$ minutes; or $21\frac{9}{11} + 5$, or $26\frac{9}{11}$ minutes past 4.

5. Let $x =$ distance minute-hand has passed over since m o'clock;

then, $\frac{x}{12} =$ distance hour-hand has passed over since m o'clock.

Then, $x - \frac{x}{12} = 5m$, no. of minute-spaces they are apart at m o'clock;

whence, $x = \frac{60m}{11} = 5\frac{5}{11}m$ minutes past m o'clock.

6. The hands are $5m$ spaces apart at m o'clock; hence to be n spaces apart the minute-hand must gain $5m - n$ spaces.

Hence, $x - \frac{x}{12} = 5m - n$;

$$x = \frac{12(5m - n)}{11} \text{ minutes past } m \text{ o'clock.}$$

NOTE.—The hands will also be n spaces apart after their meeting; hence the minute-hand must gain $5m + n$ spaces, and the time will be $12\left(\frac{5m+n}{11}\right)$ minutes past m o'clock.

CASE XV. (page 116.)

3. Let $x =$ the aunt's age;

and $\frac{x}{4} =$ Mary's age.

Then, $\frac{x}{4} + 20 = \frac{1}{2}(x + 20)$;

Transposing and uniting, $\frac{x}{4} = 10$, Mary's age;

whence, $x = 40$, the aunt's age.

4. Let x = age of the barn 6 years ago;
and $4x$ = age of the house 6 years ago.
Then, $4x+8=2(x+8)$;
whence, $x=4$, age of the barn 6 years ago;
then, $x+6=10$, age of barn;
and $4x+6=22$, age of house.

5. Let x = the number of years.
Then, $a+x=n(b+x)$;
whence, $x=\frac{bn-a}{1-n}$, the number of years.

6. Let x = B's age;
and mx = A's age.
Then, $mx+c=n(x+c)$;
whence, $x=\frac{c(n-1)}{m-n}$, B's age;
and $mx=\frac{mc(n-1)}{m-n}$, A's age.

MISCELLANEOUS PROBLEMS.

1. Let x = number of roses;
and $70-x$ = number of pinks.
Then, $x+\frac{70-x}{2}=3(70-x)$;
Clearing of fractions, $2x+70-x=420-6x$;
Transposing and uniting, $7x=350$;
whence, $x=50$, the number of roses;
and $70-x=20$, the number of pinks.

2. Let x = the number. Then, $5x+60=3(x+60)$; whence, $x=60$.

3. Let x = the first part; then, $\frac{3x}{2}$ = the second part; $\frac{9x}{4}$ = the third part; and $\frac{27x}{8}$ = the fourth part.

Then, $\frac{27x}{8}+\frac{9x}{4}+\frac{3x}{2}+x=130$; clearing of fractions, $27x+18x+12x+8x=1040$; whence, $65x=1040$; and $x=16$, the first part; $\frac{3x}{2}=24$, the second part; $\frac{9x}{4}=36$, the third part; $\frac{27x}{8}=54$, the fourth part.

4. Let x = number of trees; $\frac{x}{2}$ = number of apple trees; and $\frac{x}{3}$ = number of pear trees.

Then, $\frac{x}{2}+\frac{x}{3}+24=x$; whence, $x=144$, the number of trees.

5. Let x = the number. Then, $x+\frac{x}{2}-60=65-x$; whence, $x=50$, the number.

6. Let x = the value of the second purse, and $2x$ = the value of the first purse.

Then, $2x+12=5x$; whence, $x=4$, value of second purse; and $2x=8$, value of first purse.

7. Let x = number of ounces of copper, and $\frac{2x}{3}+4$ = number of ounces of zinc.

Then, $\frac{1}{2}(x+\frac{2x}{3}+4)+6=x$; or, $\frac{x}{2}+\frac{x}{3}+2+6=x$; transposing, $\frac{x}{6}=8$; whence, $x=48$; and $\frac{2x}{3}+4=36$.

8. Let x = the fortune; then, $\frac{3x}{5}$ = what remained at the end of the first year; $\frac{2}{5}$ of $\frac{3x}{5}=\frac{6x}{25}$ = what remained at the end of the second year.

Then, $\frac{6x}{25}=6000$; whence, $\frac{x}{25}=1000$; and $x=25{,}000$, the fortune.

9. Let x = the number of girls; and $3x$ = the number of boys.

Then, $5x+9x=210$; whence, $x=15$, the number of girls; and $3x=45$, the number of boys.

10 Let x = the time; then, $\frac{x}{4}$ = what one pipe pours in; and $\frac{x}{6}$ = what the other pipe pours in; and $\frac{x}{8}$ = what leaks out.

Then, $\frac{x}{4}+\frac{x}{6}-\frac{x}{8}=1$; whence, $6x+4x-3x=24$; and $x=3\frac{3}{7}$ hours.

11. Let x = the price of one nutmeg. Then, $3x-1=4x-2\frac{1}{2}$; whence $x=1\frac{1}{2}d.$, the price of one nutmeg.

12. Let x = the sum. Then, $x-25=2(x-60)$; whence, $x=95$, the sum.

13. Let x = the number of cows; and $\frac{x}{12}$ = the cost of keeping them.

Then, $30x = 24x + \frac{x}{12} + 142$; transposing, $\frac{71}{12}x = 142$; whence, $\frac{x}{12} = 2$; or, $x = 24$, the number of cows.

14. Let x = the number. Then, $2x - 16 - 100 = 100 - x$; whence, $3x = 216$; and $x = 72$, the number.

15. Let x = the number. Then, $\frac{x}{4} + \frac{x}{5} = \frac{x}{6} + \frac{x}{8} + 19$; clearing of fractions, $30x + 24x = 20x + 15x + 19 \times 120$; collecting, $19x = 19 \times 120$; whence, $x = 120$, the number.

16. Let x = the original quantity. Then, $\frac{3x}{4} - 24 = \frac{x}{2}$; whence, $\frac{x}{4} = 24$; and $x = 96$, the original quantity.

17. Let x = amount taken from B; and $2x$ = amount taken from A; then, $500 - x$ = amount left B; and $500 - 2x$ = amount left A.

Then, $500 - x = 3(500 - 2x)$; expanding, $500 - x = 1500 - 6x$; whence, $5x = 1000$; and $x = 200$, amount taken from B; also, $2x = 400$, amount taken from A.

18. Let x = number of pounds in the mixture; then, $\frac{x}{2} - 16$ = number of pounds of copper; $\frac{x}{3} - 12$ = number of pounds of tin; and $\frac{x}{4} + 4$ = number of pounds of lead.

Then, $\frac{x}{2} - 16 + \frac{x}{3} - 12 + \frac{x}{4} + 4 = x$; whence, $\frac{x}{12} = 24$; and $x = 288$, whole number of pounds. $\frac{x}{2} - 16 = 128$, number of pounds of copper; $\frac{x}{3} - 12 = 84$, number of pounds of tin; $\frac{x}{4} + 4 = 76$, number of pounds of lead.

19. Let x = number of idle days; and $2x$ = number of working days; then, $8x$ = what he earned; and x = what he forfeited.

Then, $8x - x = 140$; whence, $x = 20$, number of idle days.

20. Let x = A's contribution; and $2x$ = B's contribution; then, $6x$ = C's contribution; and $16x$ = D's contribution.

Then, $x + 2x + 6x + 16x = 750$; whence, $25x = 750$; and $x = 30$, A's contribution; $2x = 60$, B's contribution; $6x = 180$, C's contribution; $16x = 480$, D's contribution.

21. Let x = the sum borrowed; then, $\frac{6x}{100}$ = the interest for one year.

Then, $12\left(\frac{6x}{100}\right)+140=x$; or, $72x+14000=100x$; whence, $28x=14000$; and $x=500$, the sum borrowed.

22. Let x = the number of sheep; then, $\frac{x}{2}+12$ = the number of cows; $\frac{1}{2}\left(\frac{x}{2}+12\right)+12$ = the number of horses.

Then, $x+\frac{x}{2}+12+\frac{1}{2}\left(\frac{x}{2}+12\right)+12=128$; collecting, $\frac{7x}{4}=98$; whence, $x=56$, number of sheep; $\frac{x}{2}+12=40$, number of cows; $\frac{1}{2}\left(\frac{x}{2}+12\right)+12=32$, number of horses.

23. Let $x-2$ = the first part; $x+2$ = the second part; $\frac{x}{2}$ = the third part; and $2x$ = the fourth part.

Then, $x-2+x+2+\frac{x}{2}+2x=90$; collecting, $4\frac{1}{2}x=90$; whence, $x=20$; then, $x-2=18$, the first part; $x+2=22$, the second part; $\frac{x}{2}=10$, the third part; $2x=40$, the fourth part.

24. Let x = the side of first square; and $x+1$ = the side of the second square. Then, $x^2+44=x^2+2x+1-225$; transposing, $2x=268$; whence, $x=134$, the side of first square; $x^2+44=18{,}000$ men.

25. Let x = the sum of money. Then, $3(x-60+20=(x-20+60)$; collecting, $2x=160$; whence, $x=80$, the sum of money.

26. Let x = the number of eggs; then, $\frac{x}{4}$ = the cost of the first quantity· and $\frac{x}{6}$ = the cost of the second quantity.

Then, $\frac{x}{4}+\frac{x}{6}=\frac{2x}{5}+1$; clearing of fractions, $15x+10x=24x+60$· whence, $x=60$, the number of eggs.

27. Let x = the age of the youngest; then, $x+2\frac{1}{2}$ = the age of the second; and $x+5$ = the age of the third.

Then, $x+x+2\frac{1}{2}+x+5=43\frac{1}{2}$; collecting, $3x=36$; whence, $x=12$, the age of the youngest; $x+2\frac{1}{2}=14\frac{1}{2}$, the age of the second; $x+5=17$, the age of the oldest.

28. Let x = the whole number of votes; then, $\frac{x}{2}$ = A's votes; $\frac{x}{3}$ = B's votes; and $x-\frac{x}{2}-\frac{x}{3}=\frac{x}{6}$ = C's votes.

Then, $\frac{x}{2}-\frac{x}{6}=800$; or, $\frac{x}{3}=800$; whence, $x=2400$; $\frac{x}{2}=1200$, A's number of votes; $\frac{x}{3}=800$, B's number of votes; $\frac{x}{6}=400$, C's number of votes.

29. Let x = B's money; then, $2x$ = A's money.

Then, $\frac{4x}{5}+600=2\left(\frac{8x}{5}-600\right)$; collecting, $\frac{12x}{5}=1800$; whence, $x=750$, B's money; and $2x=1500$, A's money.

30. Let x = time it takes C; then, $2x$ = time it takes B; and $4x$ = time it takes A; also, $\frac{1}{x}$ = what C does in one hour; and $\frac{1}{2x}$ = what B does in one hour; and $\frac{1}{4x}$ = what A does in one hour.

Then, $\frac{1}{x}+\frac{1}{2x}+\frac{1}{4x}=\frac{1}{24}$; clearing of fractions, $24+12+6=x$; whence, $x=42$ hours, time it takes C; $2x=84$ hours, time it takes B; $4x=168$ hours, time it takes A.

31. Let x = the pay each child receives for one day's work; then, $\frac{5x}{3}$ = the pay each woman receives for one day's work; and $\frac{5x}{2}$ = the pay each man receives for one day's work.

Then, $150x+200x+180x=424$; collecting, $530x=424$; whence, $x=\$\frac{4}{5}$, or 80 cents, each child's daily pay.

32. Let x = F's age; then, $x+9$ = E's age.

Then, $x+9-12=\frac{3}{5}(x+7)$; clearing of fractions, $5x-15=3x+21$; whence, $x=18$, F's age; and $x+9=27$, E's age.

33. Let x = length of line; then, $\frac{x}{4}+3$ = what was cut off at one end; and $\frac{x}{5}-6$ = what was cut off at the other end.

Then, $x-\frac{x}{4}-3-\frac{x}{5}+6=25$; collecting, $\frac{11x}{20}=22$; whence, $x=40$, the length of line.

34. Let x = number of dollars; and $4x$ = number of eagles.
Then, $4x-6=6(x-6)$; whence, $x=15$, number of dollars; and $4x=60$, number of eagles.

35. Let x = her age. Then, $2(x-2\frac{1}{2})=\frac{3}{2}(x+2\frac{1}{2})$; expanding, $2x-5=\frac{3x}{2}+\frac{15}{4}$; collecting, $\frac{x}{2}=\frac{35}{4}$; whence, $x=17\frac{1}{2}$ years.

36. Let x = number who received 6 pence each;
then, $20-x$ = number who received 16 pence each.

Then, $6x+16(20-x)=240$;
Expanding, $6x+320-16x=240$;
whence, $x=8$, number who received 6 pence each.

37. Let x = number of rods;
then, $\frac{450}{x}$ = pay per rod.

Then, $\left(\frac{450}{x}+1\frac{1}{5}\right)x=540$;

Expanding, $450+\frac{6x}{5}=540$,

whence, $x=75$, number of rods.

38. Let x = amount loaned at 5%;
and $8250-x$ = amount loaned at 6%;
then, $\frac{5x}{100}$ = interest on first sum for one year;
and $\frac{6(8250-x)}{100}$ = interest on second sum for one year.

Then, $\frac{5x}{100}=\frac{6(8250-x)}{100}$;
Expanding and collecting, $11x=49500$;
whence, $x=4500$, amount at 5%;
and $8250-x=3750$, amount at 6%.

39. Let x = original number.
Then, $\frac{4}{5}\left(x-\frac{x}{8}-5000+10000\right)=60000$;
Reducing, $\frac{7x}{10}=56000$;
whence, $x=80000$, the original force.

40. Let $x=$ the first number;

and $x+1=$ the second number.

Then, $\frac{x}{5}+\frac{x}{7}=\frac{x+1}{4}+\frac{x+1}{12}$;

Clearing of fractions, $84x+60x=105x+105+35x+35$;

Collecting, $4x=140$;

whence, $x=35$, the first number;

and $x+1=36$, the second number.

41. Let $x=$ distance the minute-hand goes;

and $\frac{x}{12}=$ distance the hour-hand goes.

Since the hands are opposite, the minute-hand has gained 30+15, or 45 minute-spaces.

Hence, $x-\frac{x}{12}=45$;

and $x=49\frac{1}{11}$ minutes past 3.

Let $x=$ number in one side;

and $x^2=$ whole number of men.

Then, $(x-5)^2=x^2-295$;

Expanding, $x^2-10x+25=x^2-295$;

whence, $10x=320$;

and $x=32$, number on a side;

and $x^2=1024$, whole number of men.

43. Let $x=$ number of minutes to 4 o'clock;

then, $2x=$ number of min. past 2 o'clock $\frac{3}{4}$ hr., or 45 min., ago.

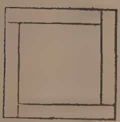
Then, $x+2x+45=120$ minutes, the time between 2 and 4 o'clock.

whence, $3x=75$;

and $x=25$ minutes of 4.

NOTE.—This solution may be illustrated by the following diagram:

44. Let $x=$ number of men in the outer rank;



then, $x-12=$ number in the side of one rectangle;

and $12(x-12)=$ number in one rectangle.

Then, $4\times 12(x-12)=1296$;

Dividing by 48, $x-12=27$;

whence, $x=39$, number in the outer rank.

NOTE.—It will be seen by the preceding figure that the regiment, arranged in a hollow square, may be divided into four equal rectangles, the longer side being equal to the number of men in the outer rank, minus 12, the depth of the ranks, and the shorter side being 12.

45. Let $x =$ the rate of current;
then, $12+x =$ the rate of going down;
and $12-x =$ rate of going up.

Then, $\dfrac{1}{12-x} = \dfrac{2}{12+x}$;

Clearing of fractions, $12+x=24-2x$;
whence, $x=4$, rate of current.

SIMPLE EQUATIONS CONTAINING TWO UNKNOWN QUANTITIES.

Art. 183. (page 123.)

6.
$$5x+2y=41; \quad (1)$$
$$3x-4y=9. \quad (2)$$

From (1) $y=\dfrac{41-5x}{2}$;

Substituting in (2), $3x-4\left(\dfrac{41-5x}{2}\right)=9$;

or, $3x-82+10x=9$;
whence, $x=7$;
and $y=\dfrac{41-35}{2}=3$.

7.
$$5x-y=48; \quad (1)$$
$$x+5y=46. \quad (2)$$

From (2), $x=46-5y$; (3)
Substituting in (1), $5(46-5y)-y=48$; (4)
or, $230-25y-y=48$;
whence, $y=7$;
and $x=46-35=11$.

8.
$$\tfrac{1}{2}x+\tfrac{1}{3}y=8; \quad (1)$$
$$\tfrac{1}{3}x+\tfrac{1}{2}y=7. \quad (2)$$

Clearing (1) of fractions, $3x+2y=48$; (3)
Clearing (2) of fractions, $2x+3y=42$; (4)
From (3), $x=\dfrac{48-2y}{3}$;

Substituting in (4), $2\left(\dfrac{48-2y}{3}\right)+3y=42$;

or, $96-4y+9y=126$;
whence, $y=6$;
and $x=\dfrac{48-12}{3}=12$.

9.
$$\frac{3x}{4}-\frac{5y}{2}=-9; \quad (1)$$
$$\frac{x}{2}+\frac{y}{3}=6. \quad (2)$$

Clearing (1) of fractions, $3x-10y=-36;$ (3)
Clearing (2) of fractions, $3x+2y=36;$ (4)
From (3), $3x=10y-36;$
Substituting in (4), $10y-36+2y=36;$
whence, $y=6;$
and $x=\frac{60-36}{3}=8.$

10.
$$\frac{4x}{5}+\frac{3z}{4}=21; \quad (1)$$
$$\frac{x}{3}-\frac{2z}{3}=-3. \quad (2)$$

Clearing (1) of fractions, $16x+15z=420;$ (3)
Clearing (2) of fractions, $x-2z=-9;$ (4)
From (4), $x=2z-9;$
Substituting in (3), $16(2z-9)+15z=420;$
or, $32z-144+15z=420;$
whence, $z=12;$
and $x=24-9=15.$

Art. 184. (page 125.)

7.
$$\frac{x}{3}+\frac{4z}{5}=6; \quad (1)$$
$$\frac{x}{6}+\frac{3z}{5}=4. \quad (2)$$

Clearing (1) of fractions, $5x+12z=90;$ (3)
Clearing (2) of fractions, $5x+18z=120;$ (4)
From (3), $x=\frac{90-12z}{5};$
From (4), $x=\frac{120-18z}{5};$
Comparing, $\frac{90-12z}{5}=\frac{120-18z}{5};$
whence, $z=5;$
and $x=\frac{90-60}{5}=6.$

8. $$\frac{2x}{3}-\frac{4y}{5}=-4; \qquad (1)$$

$$\frac{3x}{4}-\frac{2y}{3}=-1. \qquad (2)$$

Clearing (1) of fractions, $10x-12y=-60;$ (3)

Clearing (2) of fractions, $9x-8y=-12;$ (4)

From (3), $$x=\frac{12y-60}{10}=\frac{6y-30}{5};$$

From (4), $$x=\frac{8y-12}{9};$$

Comparing, $$\frac{6y-30}{5}=\frac{8y-12}{9};$$

Clearing of fractions, $54y-270=40y-60;$

whence, $y=15;$

and $$x=\frac{90-30}{5}=12.$$

9. $$\frac{3x-2y}{5}+3y=16; \qquad (1)$$

$$2x-\frac{2x-3y}{5}=11. \qquad (2)$$

Clearing (1) of fractions, $3x-2y+15y=80;$

or, $3x+13y=80;$ (3)

Clearing (2) of fractions, $10x-2x+3y=55;$

or, $8x+3y=55;$ (4)

From (3), $$x=\frac{80-13y}{3};$$

From (4), $$x=\frac{55-3y}{8};$$

Comparing, $$\frac{80-13y}{3}=\frac{55-3y}{8};$$

Clearing of fractions, $640-104y=165-9y;$

whence, $y=5;$

and $$x=\frac{80-65}{3}=5.$$

10. $$\frac{5x-6y}{6}+\frac{5y}{3}=13; \qquad (1)$$

$$\frac{5x}{6}-\frac{7x-4y}{3}=7 \qquad (2)$$

Clearing (1) of fractions, $5x-6y+10y=78;$

or, $5x+4y=78;$ (3)

Clearing (2) of fractions, $5x-14x+8y=42;$

or, $-9x+8y=42;$ (4)

From (3), $y=\frac{78-5x}{4};$

From (4), $y=\frac{42+9x}{8};$

Comparing, $\frac{78-5x}{4}=\frac{42+9x}{8};$

Clearing of fractions, $156-10x=42+9x;$

whence, $x=6;$

and $y=12.$

Art. 185. (page 126.)

7. $x+y=a;$ (1)

$x-y=b.$ (2)

Adding (1) and (2), $2x=a+b;$

and $x=\frac{a+b}{2};$

Subtracting (1) and (2), $2y=a-b;$

and $y=\frac{a-b}{2}.$

8. $ax+by=ab;$ (1)

$2ax+3by=\frac{5ab}{2}.$ (2)

Multiplying (1) by 2, $2ax+2by=2ab;$ (3)

Subtracting (2) from (3), $by=\frac{ab}{2};$

whence, $y=\frac{a}{2};$

and $x=\frac{b}{2}.$

9. $\frac{x+8}{4}+6y=21;$ (1)

$\frac{x+y}{3}=22\frac{1}{3}-5x.$ (2)

Clearing (1) of fractions, $x+8+24y=84;$
or, $x+24y=76;$ (3)
Clearing (2) of fractions, $x+y=67-15x;$
or, $16x+y=67;$ (4)
Multiplying (3) by 16, $16x+384y=1216.$ (5)
Subtracting (5) from (4), $383y=1149;$
whence, $y=3;$
and $x=4.$

10. $\frac{2x-8}{4}-\frac{3x-4y}{6}=2;$ (1)
$\frac{3x-2y}{3}+\frac{2y+6}{3}=14.$ (2)

Clearing (1) of fractions, $6x-24-6x+8y=24;$
or, $8y=48;$
whence, $y=6;$
Clearing (2) of fractions, $3x-2y+2y+6=42;$
or, $3x=36;$
whence, $x=12.$

Art. 186. (page 127.)

1. $3x+4y=24;$ (1)
$4x+3y=25.$ (2)

Multiplying (1) by 4, $12x+16y=96;$ (3)
Multiplying (2) by 3, $12x+9y=75;$ (4)
Subtracting (4) from (3), $7y=21;$
whence, $y=3;$
and $x=4.$

2. $5x-6y=7;$ (1)
$3x+3y=24.$ (2)

From (2), $3y=24-3x;$ (3)
Substituting in (1), $5x-48+6x=7;$ (4)
whence, $x=5;$
and $y=3.$

3. $2x-6y=-18;$ (1)
$6x-7y=1.$ (2)

From (1), $2x=6y-18;$ (3)
Substituting in (2), $18y-54-7y=1;$ (4)
whence, $y=5;$
and $x=6.$

4.

	$6x-5z=\frac{7}{4};$	(1)
	$-5x+4z=-\frac{3}{2}.$	(2)
Multiplying (1) by 4,	$24x-20z=7;$	(3)
Multiplying (2) by 5,	$-25x+20z=-\frac{15}{2};$	(4)
Adding (3) and (4),	$-x=-\frac{1}{2};$	
or,	$x=\frac{1}{2};$	
and	$z=\frac{1}{4}.$	

5.

	$x+y=2a;$	(1)
	$x-y=2b.$	(2)
Adding (1) and (2),	$2x=2a+2b;$	
whence,	$x=a+b;$	
and	$y=a-b.$	

6.

	$x+y=a+b;$	(1)
	$x-y=a-b.$	(2)
Adding (1) and (2),	$2x=2a;$	
whence,	$x=a;$	
and	$y=b.$	

7.

	$\frac{1}{2}x+\frac{1}{3}y=7;$	(1)
	$\frac{1}{3}x+\frac{1}{4}y=5.$	(2)
Multiplying (1) by 2,	$x+\frac{2}{3}y=14;$	(3)
Multiplying (2) by 3,	$x+\frac{3}{4}y=15.$	(4)
Subtracting (3) from (4),	$\frac{y}{12}=1;$	
whence,	$y=12;$	
and	$x=6.$	

8.

	$x+\frac{1}{4}y=14;$	(1)
	$\frac{1}{3}x+y=12.$	(2)
Multiplying (1) by 4,	$4x+y=56;$	(3)
Subtracting (2) from 3,	$3\frac{2}{3}x=44;$	(4)
Clearing of fractions,	$11x=132;$	
whence,	$x=12;$	
and	$y=8.$	

9.

	$\frac{a}{2}x+\frac{b}{3}y=2ab;$	(1)
	$\frac{a}{2}x-by=-2ab.$	(2)
Subtracting (1) from (2),	$\frac{4by}{3}=4ab;$	
whence,	$y=3a;$	
and	$x=2b.$	

10. $$6x-7y=42; \quad (1)$$
$$.7x-.6y=7.5. \quad (2)$$

Multiplying (1) by 7, $42x-49y=294;$ (3)

Multiplying (2) by 60, $42x-36y=450.$ (4)

Subtracting (3) from (4), $13y=156;$

whence, $y=12;$

and $x=21.$

11. $$\frac{x+1}{y-1}-\frac{x-1}{y}=\frac{6}{y}; \quad (1)$$
$$x-y=1. \quad (2)$$

From (2), $x=y+1.$

Substituting in (1), $\dfrac{y+2}{y-1}-\dfrac{y}{y}=\dfrac{6}{y};$

Clearing of fractions, $y^2+2y-y^2+y=6y-6;$

Collecting, $3y=6;$

whence, $y=2;$

and $x=3.$

12. $$\frac{x+y}{3}+x=15; \quad (1)$$
$$\frac{x-y}{5}+y=6. \quad (2)$$

Clearing (1) of fractions, $x+y+3x=45;$

or, $4x+y=45;$ (3)

Clearing (2) of fractions, $x-y+5y=30;$

or, $x+4y=30;$ (4)

From (4), $x=30-4y;$

Substituting in (3), $4(30-4y)+y=45;$

Expanding, $120-16y+y=45;$

whence, $y=5;$

and $x=10.$

13. $$\frac{1}{x}+\frac{1}{y}=a; \quad (1)$$
$$\frac{1}{x}-\frac{1}{y}=b. \quad (2)$$

Adding (1) and (2), $\dfrac{2}{x}=a+b;$

whence, $x=\dfrac{2}{a+b};$

and $y=\dfrac{2}{a-b}.$

14.
$$bx + ay = 2ab; \quad (1)$$
$$x + y = a + b. \quad (2)$$

Multiplying (2) by b, $bx + by = ab + b^2$; (3)
Subtracting (3) from (1), $(a-b)y = ab - b^2 = (a-b)b$;
whence, $y = b$;
and $x = a$.

15.
$$ax + by = 2m; \quad (1)$$
$$ax - by = 2n. \quad (2)$$

Subtracting (1) from (2), $2by = 2m - 2n$;
whence, $y = \frac{m-n}{b}$;
and $x = \frac{m+n}{a}$.

16.
$$ax + by = c; \quad (1)$$
$$bx + ay = d. \quad (2)$$

Multiplying (1) by b, $abx + b^2y = bc$; (3)
Multiplying (2) by a, $abx + a^2y = ad$. (4)
Subtracting (3) from (4), $(a^2 - b^2)y = ad - bc$;
whence, $y = \frac{ad - bc}{a^2 - b^2}$;
and $x = \frac{ac - bd}{a^2 - b^2}$;

17.
$$\frac{x}{a} + \frac{y}{b} = 2; \quad (1)$$
$$bx - ay = 0. \quad (2)$$

Clearing (1) of fractions, $bx + ay = 2ab$; (3)
From (2), $bx = ay$;
Substituting in (3), $2ay = 2ab$;
whence, $y = b$;
and $x = a$.

18.
$$ax + by = a; \quad (1)$$
$$bx - ay = b. \quad (2)$$

Multiplying (1) by b, $abx + b^2y = ab$; (3)
Multiplying (2) by a, $abx - a^2y = ab$. (4)
Subtracting (4) from (3), $b^2y + a^2y = 0$;
whence, $y = 0$;
and $x = 1$.

19.

$$\frac{x}{a}+\frac{y}{b}=1; \quad (1)$$

$$\frac{x}{b}+\frac{y}{a}=1. \quad (2)$$

Clearing (1) of fractions, $bx+ay=ab$; (3)

Clearing (2) of fractions, $ax+by=ab$. (4)

Subtracting (3) from (4), $ax-bx+by-ay=0$;

or, $ax-bx=ay-by$ (5);

whence, $x=y$;

Substituting in (3), $by+ay=ab$;

whence, $y=\dfrac{ab}{a+b}$;

and $x=\dfrac{ab}{a+b}$.

20.

$$\frac{a}{x}+\frac{b}{y}=m; \quad (1)$$

$$\frac{c}{x}+\frac{d}{y}=n; \quad (2)$$

Multiplying (1) by c, $\dfrac{ac}{x}+\dfrac{bc}{y}=mc$; (3)

Multiplying (2) by a, $\dfrac{ac}{x}+\dfrac{ad}{y}=na$; (4)

Subtracting (4) from (3), $\dfrac{bc-ad}{y}=mc-na$;

whence, $y=\dfrac{bc-ad}{mc-na}$;

and $x=\dfrac{bc-ad}{nb-md}$.

21.

$$(a+c)x-by=bc; \quad (1)$$

$$x+y=a+b. \quad (2)$$

Multiplying (2) by b, $bx+by=ab+b^2$; (3)

Adding (3) and (1), $(a+c+b)x=b(c+a+b)$;

whence, $x=b$;

and $y=a$.

Art. 189. CASE I. (page 129.)

2. Let x = the wages of one man;
and y = the wages of one boy.

Then, by first condition, $8x+6y=36$; (1)
and by second condition, $6x+11y=40$; (2)
Multiplying (1) by 3, $24x+18y=108$; (3)
Multiplying (2) by 4, $24x+44y=160$; (4)
Subtracting (3) from (4), $26y=52$;
whence, $y=2$, wages of one boy;
and $x=3$, wages of one man.

3. Let x = price of 1 orange;
and y = price of 1 lemon.

Then, by first condition, $12x+13y=114$; (1)
and by second condition, $6x+\frac{22y}{3}=62$. (2)
Multiplying (2) by 2, $12x+\frac{44y}{3}=124$; (3)
Subtracting (1) from (3), $\frac{5y}{3}=10$;
whence, $y=6$;
and $x=3$.

4. Let x = wages of 1 man;
and y = wages of 1 boy.

Then, by first condition, $ax+by=m$; (1)
and by second condition, $cx+dy=n$. (2)
Multiplying (1) by d, $adx+bdy=md$; (3)
and (2) by b, $bcx+bdy=nb$. (4)
Subtracting (4) from (3), $(ad-bc)x=md-nb$;
whence, $x=\frac{md-nb}{ad-bc}$;
and $y=\frac{an-mc}{ad-bc}$.

CASE II. (page 130.)

2. Let $\frac{x}{y}$ = the fraction.

Then, by the first condition, $\frac{x-2}{y}=\frac{1}{4};$ (1)

and by the second condition, $\frac{x}{y-2}=\frac{1}{2}.$ (2)

Clearing (1) of fractions, $4x-8=y;$ (3)

Clearing (2) of fractions, $2x=y-2.$ (4)

Subtracting (4) from 3, $2x=10;$

whence, $x=5;$

and $y=12;$

therefore, $\frac{x}{y}=\frac{5}{12}.$

3. Let $\frac{x}{y}$ = the fraction.

Then, by first condition, $\frac{x-4}{y-4}=\frac{1}{3};$ (1)

and by second condition, $\frac{x+5}{y+5}=\frac{5}{6}.$ (2)

Clearing (1) of fractions, $3x-12=y-4;$ (3)

Clearing (2) of fractions, $6x+30=5y+25;$ (4)

Multiplying (3) by 2, $6x-24=2y-8;$ (5)

Subtracting (5) from (4), $54=3y+33;$

whence, $y=7;$

and $x=5;$

therefore, $\frac{x}{y}=\frac{5}{7}.$

4. Let $\frac{x}{y}$ = the fraction.

Then, by first condition, $\frac{x+1}{y+1}=\frac{1}{2}.$ (1)

and by second condition, $\frac{x-1}{x+y}=\frac{1}{4}.$ (2)

Clearing (1) of fractions and collecting, $2x+1=y;$ (3)

Clearing (2) of fractions and collecting, $3x-4=y.$ (4)

Subtracting (3) from (4), $x=5;$

and $y=11;$

whence, $\frac{x}{y}=\frac{5}{11}.$

5. Let $\dfrac{x}{y}=$ the fraction.

Then, by first condition, $\dfrac{x+a}{y}=\dfrac{m}{n}$; (1)

and by second condition, $\dfrac{x}{y+a}=\dfrac{n}{m}$. (2)

From (1), $x=\dfrac{my-an}{n}$; (3)

From (2), $x=\dfrac{ny+an}{m}$; (4)

Comparing (3) and (4), $\dfrac{my-an}{n}=\dfrac{ny+an}{m}$;

whence, $(m^2-n^2)y=amn+an^2$;

and $y=\dfrac{amn+an^2}{m^2-n^2}=\dfrac{an}{m-n}$;

and $x=\dfrac{an}{m-n}$;

Therefore, $\dfrac{x}{y}=\dfrac{an(m-n)}{an(m-n)}$.

CASE III. (page 131.)

2. Let $x=$ the tens' digit;
and $y=$ the units' digit;
then, $10x+y=$ the number;
and $10y+x=$ the number with digits inverted.

Then, by first condition, $10x+y+10y+x=132$; (1)
and by second condition, $10x+y-(10y+x)=18$. (2)

Uniting terms in (1), $11x+11y=132$; (3)
Dividing by 11, $x+y=12$; (4)
Uniting terms in (2), $9x-9y=18$; (5)
Dividing by 9, $x-y=2$; (6)
Uniting (4) and (6), $x=7$ and $y=5$;
whence, $10x+y=75$.

3. Let $x=$ the tens' digit;
and $y=$ the units' digit;
then, $10x+y=$ the number.

Then, by first condition, $\dfrac{10x+y}{x+y}=4$; (1)

and by second condition, $10x+y+36=10y+x$. (2)

Clearing (1) of fractions, $10x+y=4x+4y$; (3)
Transposing and uniting, $6x=3y$;
whence, $2x=y$; (4)
Transposing and uniting (2), $9y-9x=36$; (5)
Dividing by 9, $y-x=4$; (6)
Substituting from (4), $x=4$ and $y=8$;
whence, $10x+y=48$.

4. Let $x=$ the tens' digit;
and $y=$ the units' digit;
then, $10x+y=$ the number.

Then, by first condition, $10x+y=5x+5y$; (1)
and by second condition, $20x+2y-(3x+3y+9)=10y+x$. (2)

Transposing and uniting (1), $5x=4y$; (3)
Transposing and uniting (2); $16x-11y=9$; (4)
Substituting from (3), $\dfrac{64y}{5}-11y=9$; (5)
Clearing (5) of fractions, $64y-55y=45$;
whence, $y=5$ and $x=4$;
therefore, $10x+y=45$.

CASE IV. (page 132.)

2. Let $x=$ the principal;
and $y=$ the time.

Then, by first condition, $xy\times\frac{6}{100}+x=310$; (1)
and by second condition, $xy\times\frac{10}{100}+x=350$. (2)

Subtracting (1) from (2), $\dfrac{4xy}{100}=40$;
whence, $xy=1000$; (3)
Substituting (3) in (1), $60+x=310$; (4)
whence, $x=250$;
and $y=4$.

3. Let $x=$ the principal;
and $y=$ the rate.

Then, by first condition, $\dfrac{7xy}{100}+x=810$; (1)

and by second condition, $\dfrac{12xy}{100}+x=960$. (2)

Subtracting (1) from (2), $\frac{5xy}{100}=150;$

and $xy=3000;$ (3)

Substituting (3) in (1), $210+x=810;$ (4)

whence, $x=600;$

and $y=5.$

4. Let $x=$ the principal;

and $y=$ the rate.

Then, by first condition, $\frac{mxy}{100}+x=a;$ (1)

and by second condition, $\frac{nxy}{100}+x=b.$ (2)

Subtracting (1) from (2), $(n-m)\frac{xy}{100}=b-a;$

and $xy=\frac{100(b-a)}{n-m};$ (3)

whence, $x=\frac{an-bm}{n-m}$ and $y=\frac{100(b-a)}{an-bm}.$

CASE V. (page 133.)

2. Let $x=$ the greater number;

and $y=$ the less number.

Then, by first condition, $x+y=3xy;$ (1)

and by second condition, $x-y=xy.$ (2)

Adding (1) and (2), $2x=4xy$

Dividing by $2x$, $2y=1;$

whence, $y=\frac{1}{2}:$

and $x=1.$

3. Let $x=$ the greater number;

and $y=$ the less number.

Then, by first condition, $2x+2y=3xy;$ (1)

and by second condition, $2x-2y=xy.$ (2)

Adding (1) and (2), $4x=4xy;$

whence, $y=1;$

and $x=2.$

4\. Let $x =$ the greater number:
and $y =$ the less number.

Then, by first condition, $x+y=\frac{4x}{y}$; (1)

and by second condition, $x-y=\frac{2x}{y}$. (2)

Adding (1) and (2), $2x=\frac{6x}{y}$;

Dividing by $2x$, $1=\frac{3}{y}$;

whence, $y=3$;
and $x=9$.

5\. Let $x =$ the greater number;
and $y =$ the less number.

Then, by first condition, $x+y=axy$; (1)
and by second condition, $x-y=bxy$. (2)
Adding (1) and (2), $2x=(a+b)xy$;

Dividing by $(a+b)x$, $y=\frac{2}{a+b}$;

and $x=\frac{2}{a-b}$.

MISCELLANEOUS PROBLEMS.

1\. Let $x =$ price of tea per pound;
and $y =$ price of sugar per pound.
Then, by first condition, $8x+3y=264$; (1)
and by second condition, $5x+4y=182$; (2)
Multiplying (1) by 4, $32x+12y=1056$; (3)
and (2) by 3, $15x+12y=546$; (4)
Subtracting (4) from (3), $17x=510$.
whence, $x=30$;
and $y=8$.

2. Let $x =$ the greater number;

and $y =$ the less number.

Then, by first condition, $x + \frac{1}{3}y = 30$; (1)

and by second condition, $y \div \frac{x}{5} = 3$. (2)

Clearing (1) of fractions, $3x + y = 90$; (3)

Reducing (2), $y = \frac{3x}{5}$; (4)

Substituting (4) in (3), $3x + \frac{3x}{5} = 90$;

Dividing by 3 and uniting, $\frac{6x}{5} = 30$;

whence, $x = 25$;

and $y = 15$.

3. Let $x =$ what A had;

and $y =$ what B had.

Then, by first condition, $x + 200 = 3(y - 200)$; (1)

and by second condition, $y + 200 = 2(x - 200)$. (2)

Transposing and uniting (1), $3y - x = 800$; (3)

Transposing and uniting (2), $y - 2x = -600$; (4)

Multiplying (3) by 2, $6y - 2x = 1600$. (5)

Subtracting (4) from (5), $5y = 2200$;

whence, $y = 440$;

and $x = 520$.

4. Let $x =$ value of gold watch;

and $y =$ value of silver watch.

Then, by first condition, $y + 20 = \frac{x}{3}$; (1)

and by second condition, $x + 20 = 7y$. (2)

Clearing (1) of fractions, $3y + 60 = x$; (3)

Subtracting (3) from (2), $4y = 80$;

whence, $y = 20$;

and $x = 120$.

5. Let $\frac{x}{y}$ = the fraction.

Then, by first condition, $\frac{x+1}{y}=\frac{1}{3}$; (1)

and by second condition, $\frac{x}{y+1}=\frac{1}{4}$. (2)

Clearing (1) of fractions, $3x+3=y$; (3)
Clearing (2) of fractions, $4x=y+1$; (4)
Subtracting (3) from (4), $x=4$;
and $y=15$;

whence, $\frac{x}{y}=\frac{4}{15}$.

6. Let x = Mary's number;
and y = Jane's number.
Then, by first condition, $x+4=3y$; (1)

and by second condition, $x-4=\frac{y}{3}$. (2)

Subtracting (2) from (1), $\frac{8y}{3}=8$;

whence, $y=3$;
and $x=5$.

7. Let x = the number of guineas;
and y = the number of moidores.
Then, by first condition, $21x+27y=2400$; (1)
and by second condition, $x+y=100$. (2)
Multiplying (2) by 21, $21x+21y=2100$; (3)
Subtracting from (1), $y=50$;
and $x=50$.

8. Let x = the number of apples;
and y = the number of pears.

Then, by first condition, $\frac{x}{4}+\frac{y}{5}=30$; (1)

and by second condition, $\frac{x}{8}+\frac{y}{15}=13$. (2)

Multiplying (2) by 2, $\frac{x}{4}+\frac{2y}{15}=26$; (3)

Subtracting (3) from (1), $\frac{y}{15}=4$;

whence, $y=60$;
and $x=72$.

9. Let x = number of acres of the better sort;
and y = number of acres of the poorer sort.

Then, by first condition, $x+y=25$; (1)
and by second condition, $8x+5y=152$. (2)

Multiplying (1) by 5, $5x+5y=125$; (3)
Subtracting (3) from (2), $3x=27$;
whence, $x=9$;
and $y=16$.

10. Let x = the tens digit;
and y = the units' digit;
then, $10x+y$ = the number.

Then, by first condition, $10x+y+9=10y+x$; (1)
and by second condition, $10x+y+10y+x=33$. (2)

Transposing and uniting (1), $9y-9x=9$;
Dividing by 9, $y-x=1$; (3)
Transposing and uniting (2), $11x+11y=33$;
Dividing by 11, $x+y=3$; (4)
Adding (3) and (4), $y=2$;
and $x=1$;
whence, $10x+y=12$.

11. Let x = A's money;
and y = B's money.
Then, by first condition, $x+20=2(y-20)$; (1)
and by second condition, $y+20=3(x-20)$. (2)
Transposing and uniting (1), $2y-x=60$; (3)
Transposing and uniting (2), $3x-y=80$; (4)
Multiplying (3) by 3, $-3x+6y=180$. (5)
Adding (4) and (5), $5y=260$;
whence, $y=52$;
and $x=44$.

12. Let x = A's money;
and y = B's money.
Then, by first condition, $x-20=y+20$; (1)

and by second condition, $y-20=\frac{x+20}{2}$; (2)

Subtracting (2) from (1), $\frac{x-60}{2}=40$;

whence, $x=140$;
and $y=100$.

13. Let $x=$ the tens' digit;
and $y=$ the units' digit;
then, $10x+y=$ the number.

Then, by first condition, $10x+y=2x+2y;$ (1)
and by second condition, $50x+5y-9=10y+x.$ (2)

Uniting (1), $8x=y;$ (3)
Uniting (2), $49x-5y=9;$ (4)
Substituting (3) in (4), $49x-40x=9;$
whence, $x=1$ and $y=8;$
and $10x+y=18.$

14. Let $x=$ number of persons;
and $y=$ what each paid;
then $xy=$ whole sum paid.

Then, by first condition, $\frac{xy}{x+3}=y-1;$ (1)

and by second condition, $\frac{xy}{x-2}=y+1.$ (2)

Clearing (1) of fractions, $xy=xy-x+3y-3;$
Transposing and uniting, $3y-x=3;$ (3)
Clearing (2) of fractions, $xy=xy+x-2y-2;$
Transposing and uniting, $2y-x=-2;$ (4)
Subtracting (4) from (3), $y=5;$
and $x=12.$

15. Let $x=$ number of yards A ran in one minute;
and $y=$ number of yards B ran in one minute.

Then, since B was 20 yards ahead and A ran 30 yards beyond, A ran 30+20, or 50 yards more; hence, we have (1); and since A ran 3 yards while B ran 2, we have twice A's equal to 3 times B's, whence we obtain (2).

$5x-5y=50;$
$x-y=10;$ (1)
$2x=3y;$
$x=\frac{3}{2}y;$ (2)
$\frac{3}{2}y-y=10;$
$y=20;$
$x=30;$
$5x=150$, the whole course.

NOTE.—This can be solved by using one unknown quantity, letting $3x$ and $2x$ equal the number of yards each ran a minute. $15x-10x=50$; whence, $x=10$, $3x=30$, and $2x=20$.

16. Let x = the length of the field;
and y = the breadth of the field;
then, xy = the area.

Then, by first condition, $(x+10)(y+5)=xy+400;$ (1)
and by second condition, $(x+5)(y+10)=xy+450.$ (2)
Expanding (1), $xy+5x+10y+50=xy+400;$
Uniting, $5x+10y=350;$
Dividing by 5, $x+2y=70;$ (3)
Expanding (2), $xy+10x+5y+50=xy+450;$
Uniting, $10x+5y=400;$
Dividing by 5, $2x+y=80;$ (4)
Multiplying (3) by 2, $2x+4y=140.$ (5)
Subtracting (4) from (5), $3y=60;$
whence, $y=20$, and $x=30$.

17. Let x = the number of first kind;
and y = the number of second kind.

Then, since one of the first kind costs $\frac{7}{3}$ of a cent and one of the second $\frac{5}{2}$ of a cent, we have (1). She sold them for 36 cents a dozen, or 3 cents apiece, and gained 62 cents, therefore they were sold for 324 cents; hence, we have (2). Clearing (1) of fractions and multiplying (2) by 5, and subtracting, we find $x=48$, $y=60$.

$$\frac{7x}{3}+\frac{5y}{2}=262; \quad (1)$$
$$3x+3y=324. \quad (2)$$
$$14x+15y=1572; \quad (3)$$
$$15x+15y=1620. \quad (4)$$
$$x=48;$$
$$y=60.$$

18. Let x = the number of yards A runs in one second;
and y = the number of yards B runs in one second;
then, $\frac{3600x}{1760}$ = the number of miles A runs in one hour.

Since A runs x yards in one second, to run one yard requires $\frac{1}{x}$ of a second, and to run 1760 yards, or one mile, $\frac{1760}{x}$ seconds. Since B has a start of 20 yards, he runs $1760-20$, or 1740 yards after both start, and requires $\frac{1740}{y}$ seconds, which, by the first condition, is 30 seconds longer than A requires; hence we have (1). In the second heat B runs 32 seconds longer than A, or

$$\frac{1760}{x}+30=\frac{1740}{y}; \quad (1)$$
$$\frac{1760}{x}+32=\frac{1750\frac{6}{11}}{y}. \quad (2)$$
$$2=\frac{10\frac{6}{11}}{y}; \quad (3)$$
$$y=5\tfrac{3}{11};$$
$$\frac{1760}{x}+30=\frac{1740}{5\frac{3}{11}}=330;$$
$$300x=1760;$$
$$x=\frac{176}{30};$$
$$\frac{3600x}{1760}=12.$$

$\dfrac{1760}{x}+32$ seconds, and since he runs $9\frac{5}{11}$ yards less, he runs $1760-9\frac{5}{11}$, or $1750\frac{6}{11}$ yards; and if he runs y yards in one second, the number of seconds is $\dfrac{1750\frac{6}{11}}{y}$; whence we have (2). Subtracting (1) from (2) we have $y=5\frac{3}{11}$, whence by substituting in (1) we have $x=\dfrac{176}{30}$; and by substituting the value of x, we have $\dfrac{3600x}{1760}=12$. *Ans.*

EQUATIONS CONTAINING THREE OR MORE UNKNOWN QUANTITIES.

Art. 190. (page 136.)

2.

	$x+2y+z=13;$	(1)
	$x+y+3z=15;$	(2)
	$2x+3y+2z=22.$	(3)
Subtracting (1) from (2),	$-y+2z=2;$	(4)
Multiplying (1) by 2,	$2x+4y+2z=26;$	(5)
Subtracting (3) from (5),	$y=4;$	
Substituting in (4),	$z=3;$	
Substituting in (1),	$x=2.$	

3.

	$2x+4y+3z=35;$	(1)
	$x+3y+2z=23;$	(2)
	$3x-5y+4z=17.$	(3)
Multiplying (2) by 2,	$2x+6y+4z=46;$	(4)
Subtracting (1) from (4),	$2y+z=11;$	(5)
Multiplying (2) by 3,	$3x+9y+6z=69;$	(6)
Subtracting (3) from (6),	$14y+2z=52;$	(7)
Dividing (7) by 2,	$7y+z=26;$	(8)
Subtracting (5) from (8),	$5y=15;$	
whence,	$y=3;$	
and	$z=5;$	
and	$x=4.$	

4.

	$2x+4y-3z=22;$	(1)
	$4x-2y+5z=18;$	(2)
	$6x+7y-z=63.$	(3)
Multiplying (1) by 2,	$4x+8y-6z=44;$	(4)
Subtracting (2) from (4),	$10y-11z=26;$	(5)
Multiplying (1) by 3,	$6x+12y-9z=66;$	(6)
Subtracting (3) from (6),	$5y-8z=3;$	(7)
Multiplying (7) by 2,	$10y-16z=6;$	(8)
Subtracting (8) from (5),	$5z=20;$	
whence,	$z=4;$	
and	$y=7;$	
and	$x=3.$	

5.

	$2x+3y-z=27;$	(1)
	$3x-4y+3z=12;$	(2)
	$4x+2y-5z=15.$	(3)
Multiplying (1) by 2,	$4x+6y-2z=54;$	(4)
Subtracting (3) from (4),	$4y+3z=39;$	(5)
Multiplying (1) by 3,	$6x+9y-3z=81;$	(6)
Multiplying (2) by 2,	$6x-8y+6z=24.$	(7)
Subtracting (7) from (6),	$17y-9z=57;$	(8)
Multiplying (5) by 3,	$12y+9z=117.$	(9)
Adding (8) and (9),	$29y=174;$	
whence,	$y=6;$	
and	$z=5;$	
and	$x=7.$	

6.

	$x+\frac{1}{2}y+\frac{1}{3}z=32;$	(1)
	$\frac{1}{3}x+\frac{1}{4}y+\frac{1}{5}z=15;$	(2)
	$\frac{1}{4}x+\frac{1}{5}y+\frac{1}{6}z=12.$	(3)
Clearing (1) of fractions,	$6x+3y+2z=192;$	(4)
Clearing (2) of fractions,	$20x+15y+12z=900;$	(5)
Clearing (3) of fractions,	$15x+12y+10z=720;$	(6)
Multiplying (4) by 6,	$36x+18y+12z=1152;$	(7)
Subtracting (5) from (7),	$16x+3y=252;$	(8)
Multiplying (4) by 5,	$30x+15y+10z=960;$	(9)
Subtracting (6) from (9),	$15x+3y=240;$	(10)
Subtracting (10) from (8),	$x=12;$	
whence,	$y=20;$	
and	$z=30.$	

7.

$$x+y+z=24; \quad (1)$$
$$x-y+z=8; \quad (2)$$
$$x+y-z=6. \quad (3)$$

Subtracting (2) from (1), $y=8$;
Subtracting (3) from (1), $z=9$;
Adding (2) and (3), $x=7$.

8.

$$x+y+z=a; \quad (1)$$
$$x+y-z=b; \quad (2)$$
$$x-y+z=c. \quad (3)$$

Adding (2) and (3), $x=\frac{1}{2}(b+c)$;
Subtracting (2) from (1), $z=\frac{1}{2}(a-b)$;
Subtracting (3) from (1), $y=\frac{1}{2}(a-c)$.

9.

$$x+y=a; \quad (1)$$
$$x+z=b; \quad (2)$$
$$y+z=c. \quad (3)$$

Adding (1), (2) and (3), $2x+2y+2z=a+b+c$; (4)
Multiplying (1) by 2, $2x+2y=2a$ (5)
Subtracting (5) from (4), $z=\frac{1}{2}(b+c-a)$;
Multiplying (2) by 2, $2x+2z=2b$; (6)
Subtracting (6) from (4), $y=\frac{1}{2}(a+c-b)$;
Multiplying (3) by 2, $2y+2z=2c$ (7)
Subtracting (7) from (4), $x=\frac{1}{2}(a+b-c)$.

This example may also be solved as follows:

Subtracting (2) from (1), $y-z=a-b$; (4)
Adding (3) and (4), $y=\frac{1}{2}(a+c-b)$;
Subtracting (4) from (3), $z=\frac{1}{2}(b+c-a)$;
Substituting y in (1), $x+\frac{1}{2}(a+c-b)=a$;
whence, $x=\frac{1}{2}(a+b-c)$.

10. Given $\frac{1}{x}+\frac{1}{y}=5$, (1); $\frac{1}{x}+\frac{1}{z}=6$, (2); $\frac{1}{z}+\frac{1}{y}=7$, (3). Subtracting (1) from (2), $\frac{1}{z}-\frac{1}{y}=1$, (4); adding (3) and (4), $\frac{2}{z}=8$; whence, $z=\frac{1}{4}$; subtracting (4) from (3), $\frac{2}{y}=6$; whence, $y=\frac{1}{3}$; and $x=\frac{1}{2}$.

11. Given $\frac{x}{a}+\frac{y}{b}=1$, (1); $\frac{x}{a}+\frac{z}{c}=1$, (2); $\frac{y}{b}+\frac{z}{c}=1$, (3). Subtracting (2) from (1), $\frac{y}{b}-\frac{z}{c}=0$, (4); hence, $\frac{y}{b}=\frac{z}{c}$, (5); substituting (5) in (3), $\frac{2z}{c}=1$; hence, $\frac{z}{c}=\frac{1}{2}$; and $z=\frac{c}{2}$; also, $\frac{y}{b}=\frac{1}{2}$; and $y=\frac{b}{2}$; substituting, $\frac{x}{a}=\frac{1}{2}$; and $x=\frac{a}{2}$.

NOTE.—This problem may also be solved by adding the three equations, dividing by 2, and then subtracting each equation from the result.

12.

$$\begin{array}{lr} x+y-z=c; & (1)\\ x+z-y=b; & (2)\\ y+z-x=a. & (3)\end{array}$$

Adding (1) and (2), $x=\frac{1}{2}(b+c)$;
Adding (2) and (3), $z=\frac{1}{2}(a+b)$;
Adding (1) and (3), $y=\frac{1}{2}(a+c)$.

13.

$$\begin{array}{lr} x+y+z=12; & (1)\\ x+y+u=13; & (2)\\ x+z+u=14; & (3)\\ y+z+u=15. & (4)\end{array}$$

If s equal the sum of the four quantities, the equations will be (5), (6), (7), and (8).

$$\begin{array}{lr} s-u=12; & (5)\\ s-z=13; & (6)\\ s-y=14; & (7)\\ s-x=15. & (8)\end{array}$$

Adding the equations, $4s-x-y-z-u=54$;
or, $4s-s=54$;
whence, $s=18$; $u=6$; $z=5$; $y=4$; $x=3$.

This example may also be solved as follows:

Adding the four equations, $3x+3y+3z+3u=54$; whence, $x+y+z+u=18$, (5); subtracting (1) from (5), $u=6$; subtracting (2) from (5), $z=5$; subtracting (3) from (5), $y=4$; subtracting (4) from 5, $x=3$.

14.

$$\begin{array}{lr} \frac{a}{x}+\frac{b}{y}+\frac{c}{z}=3; & (1)\\ \frac{a}{x}-\frac{b}{y}+\frac{c}{z}=1; & (2)\\ \frac{a}{x}+\frac{2b}{y}-\frac{c}{z}=2. & (3)\end{array}$$

Subtracting (2) from (1), $\frac{2b}{y}=2$; and $y=b$; adding (2) and (3), $\frac{2a}{x}+\frac{b}{y}=3$; substituting, $\frac{2a}{x}=2$; whence, $x=a$; and $z=c$.

PROBLEMS PRODUCING EQUATIONS CONTAINING THREE UNKNOWN QUANTITIES.

1. Let x = the number of bushels of wheat;
y = the number of bushels of rye;
and z = the number of bushels of oats.

Then, $x+y+z=47$; (1)
and $x+y-z=7$; (2)
and $x+z-y=17$. (3)
Adding (2) and (3), $x=12$;
Subtracting (2) from (1), $z=20$;
Subtracting (3) from (1), $y=15$.

2. Let x = A's fortune;
y = B's fortune;
and z = C's fortune.

Then, $\frac{x}{2}+\frac{y}{3}+\frac{z}{4}=600$; (1)
and $4x+3y+2z=5000$; (2)
also, $x+y+z=1800$. (3)
Clearing (1) of fractions, $6x+4y+3z=7200$; (4)
Multiplying (3) by 3, $3x+3y+3z=5400$. (5)
Subtracting (5) from (4), $3x+y=1800$; (6)
Multiplying (3) by 2, $2x+2y+2z=3600$; (7)
Subtracting (7) from (2), $2x+y=1400$; (8)
Subtracting (8) from (6), $x=400$;
whence, $y=600$;
and $z=800$.

3. Let x = the number of horses;
y = the number of cows;
and z = the number of sheep.

Then, $x+y+z=230$; (1)
and $x+\frac{y}{2}+\frac{z}{2}=145$; (2)
and $\frac{x}{5}+y+\frac{z}{5}=110$; (3)
Clearing (2) of fractions, $2x+y+z=290$; (4)
Subtracting (1) from (4), $x=60$;
Clearing (3) of fractions, $x+5y+z=550$; (5)
Subtracting (1) from (5), $4y=320$;
whence, $y=80$;
and $z=90$.

4. Let $x=$ the first part;
$y=$ the second part;
and $z=$ the third part.

Then, $x+y+z=150$; (1)
and $2x+35=3y+5$; (2)
and $3y+5=\frac{4z}{5}$. (3)

Transposing (2), $3y-2x=30$; (4)
Multiplying (1) by 2, $2x+2y+2z=300$. (5)
Adding (4) and (5), $5y+2z=330$. (6)
Clearing (3) of fractions, $4z-15y=25$; (7)
Multiplying (6) by 2, $4z+10y=660$. (8)
Subtracting (7) from (8), $25y=635$;
and $y=25\frac{2}{5}$;
also, $x=23\frac{1}{10}$;
and $z=101\frac{1}{2}$.

5. Let $x=$ A's age;
$y=$ B's age;
and $z=$ C's age.

Then, $x+y=94$; (1)
and $y+z=98$; (2)
and $x+z=96$. (3)
Subtracting (1) from (2), $z-x=4$; (4)
Adding (3) and (4), $z=50$;
Subtracting (4) from (3), $x=46$;
Substituting, $y=48$.

6. Let $x=$ the number of ones;
$y=$ the number of fives;
and $z=$ the number of tens.

Then, $x+5y+10z=82$; (1)
and $x+\frac{5y}{2}+5z=47$; (2)
and $5y+\frac{x}{4}+\frac{10z}{4}=43$. (3)

Clearing (2) of fractions, $2x+5y+10z=94$; (4)
Subtracting (1) from (4), $x=12$;
Clearing (3) of fractions, $20y+x+10z=172$; (5)
Subtracting (1) from (5), $15y=90$;
whence, $y=6$;
and $z=4$.

7. Let x=price of one apple; y=price of one peach; z=price of one pear; and u=price of one orange.

Then, $3x+4y=11$, (1); $4x+5z=19$, (2); $4z+5u=32$, (3); $6x+7u=34$, (4)

Multiplying (2) by 4, $16x+20z=76$, (5); multiplying (3) by 5, $20z+25u=160$, (6).

Subtracting (5) from (6), $25u-16x=84$, (7); multiplying (7) by 6, $150u-96x=504$, (8); multiplying (4) by 16, $96x+112u=544$, (9).

Adding (8) and (9), $262u=1048$; whence, $u=4$; $x=1$; $y=2$; $z=3$.

8. Let x=the first part; y=the second part; z=the third part; and u=the fourth part.

Then, $x+y+z+u=27$, (1); $x+2=y-2$, (2); $y-2=2z$, (3); $2z=\frac{u}{2}$, (4).

From (2), $x=y-4$, (5); from (3), $z=\frac{y-2}{2}$, (6); from (4), $u=4z$, (7); substituting value of z from (6), $u=2y-4$, (8); substituting values of x, z, u in (1), $y-4+y+\frac{y-2}{2}+2y-4=27$, (9); transposing and uniting, $9y=72$; whence, $y=8$; $x=4$; $z=3$; $u=12$.

NOTE.—This problem may also be solved as follows:

Let $x-2$=first part; then, $x+2$=second part; and $\frac{x}{2}$=third part; and $2x$=fourth part.

Then, $x-2+x+2+\frac{x}{2}+2x=27$; whence, $x=6$; $x-2=4$; $x+2=8$; $\frac{x}{2}=3$; $2x=12$.

9. Let x=the number of days it takes A; y=the number of days it takes B; and z=the number of days it takes C.

Then, $\frac{1}{x}+\frac{1}{y}=\frac{1}{12}$, (1); $\frac{1}{x}+\frac{1}{z}=\frac{1}{15}$, (2); $\frac{1}{y}+\frac{1}{z}=\frac{1}{20}$, (3).

Subtracting (2) from (1), $\frac{1}{y}-\frac{1}{z}=\frac{1}{60}$, (4); adding (3) and (4), $\frac{2}{y}=\frac{4}{60}$; whence, $\frac{1}{y}=\frac{2}{60}$; and $y=30$; substituting, $z=60$; and $x=20$.

10. Let x = the first fraction; y = the second fraction; and z = the third fraction.

Then, $x+y+z=2\frac{1}{4}$, (1); $x+z=2y$, (2); $z-x=\frac{z}{5}$, (3).

Transposing (3), $x=\frac{4z}{5}$, (4); substituting value of x in (2), $\frac{4z}{5}+z=2y$; whence, $y=\frac{9z}{10}$, (5); substituting values of x and y in (1), $\frac{4z}{5}+\frac{9z}{10}+z=\frac{9}{4}$, (6); clearing (6) of fractions, $16z+18z+20z=45$, (7); whence, $z=\frac{5}{6}$; $x=\frac{2}{3}$; $y=\frac{3}{4}$.

11. Let x = the number sunk; y = the number burned; and z = the number captured.

Then, $x+y+z+15=8x$, (1); $y=x-2$, (2); $z=x+7$, (3).

Substituting values of y and z in (1), $x+x-2+x+7+15=8x$, (4); transposing and uniting, $5x=20$; whence, $x=4$; and $8x=32$.

This can be solved by using only one unknown quantity, as follows:

Let x = the number sunk; $x-2$ = the number burned; $x+7$ = the number captured.

Then, $x+x-2+x+7+15=8x$; whence, $x=4$; and $8x=32$.

12. Let x = the time it takes first pipe; y = the time it takes second pipe; and z = the time it takes third pipe.

Then, $\frac{1}{y}+\frac{1}{z}=\frac{1}{20}$, (1); $\frac{1}{x}+\frac{1}{z}=\frac{1}{25}$, (2); $\frac{1}{x}+\frac{1}{y}=\frac{1}{30}$, (3).

Subtracting (2) from (3), $\frac{1}{z}-\frac{1}{y}=\frac{1}{150}$, (4); adding (1) and (4), $\frac{2}{z}=\frac{17}{300}$; whence, $z=35\frac{5}{17}$; subtracting (4) from (1), $\frac{2}{y}=\frac{13}{300}$, (5); whence, $y=46\frac{2}{13}$; and $x=85\frac{5}{7}$.

13. Let x = value of first watch; y = value of second watch; and z = value of third watch.

Then, $x+60=\frac{y+z}{3}$, (1); $y+60=\frac{3}{5}(x+z)$, (2); $z+60=3(x+y)$, (3).

Clearing (1) of fractions and transposing, $y+z-3x=180$, (4); clearing (2) of fractions and transposing, $3x+3z-5y=300$, (5); expanding (3), $3x+3y-z=60$, (6).

Subtracting (6) from (5), $4z-8y=240$, (7); adding (4) and (5), $4z-4y=480$, (8).

Subtracting (7) from (8), $4y=240$; whence, $y=60$; $z=180$; and $x=20$.

14. Let x = the hundreds' digit; y = the tens' digit; and z = the units' digit; then, $100x+10y+z$ = the number.

Then, $x+y+z=9$; (1)

$x+z=2y$; (2)

$100x+10y+z+198=x+10y+100z.$ (3)

Subtracting (2) from (1), $3y=9$;

whence, $y=3$;

Transposing and uniting (3), $99z-99x=198$; (4)

Dividing (4) by 99, $z-x=2$; (5)

Substituting value of y in (2), $x+z=6.$ (6)

Adding (5) and (6), $z=4$;

whence, $x=2$;

and $100x+10y+z=234.$

15. Let $x=$ number of quarter dollars;

$y=$ number of dimes;

$z=$ number of half-dimes;

then, $25x+10y+5z=$ value in cents.

Then, $\frac{25x+10y+5z}{10}=x+y+z$; (1)

$\frac{25x+10y+5z}{25}=y$; (2)

$z-y=1.$ (3)

Clearing (1) of fractions, $25x+10y+5z=10x+10y+10z$;

Uniting the terms, $15x=5z$;

whence, $3x=z$; (4)

Clearing (2) of fractions, $25x+10y+5z=25y$; (5)

Dividing (5) by 5, $5x+z=3y$;

Substituting from (3) and (4), $\frac{5z}{3}+z=3z-3$;

whence, $z=9$;

and $y=8$;

and $x=3.$

16. Let $x+y+z=$ A's number;

$y=$ B's number;

and $z=$ C's number.

Then, $x=$ what A had after first game;

$2y=$ what B had after first game;

$2z=$ what C had after first game;

$2x=$ what A had after second game;

$2y-x-2z=$ what B had after second game;

$4z=$ what C had after second game;

$4x=$ what A had after third game;

$4y-2x-4z=$ what B had after third game;

$4z-2y+x+2z-2x=$ what C had after third game.

Then, $4x = 16;$ (1)

$4y - 2x - 4z = 16;$ (2)

$6z - 2y - x = 16.$ (3)

From (1), $x = 4;$ (4)

Substituting in (2), $4y - 4z = 24;$ (5)

Substituting in (3), $6z - 2y = 20;$ (6)

Dividing (5) by 2, $2y - 2z = 12.$ (7)

Adding (6) and (7), $4z = 32;$

whence, $z = 8;$

and $y = 14$

and $x + y + z = 26.$

17. Let $x =$ the number of bales;

and $y =$ the number of casks;

then, $\frac{1}{x} =$ the part of the cave occupied by 1 bale;

and $\frac{1}{y} =$ the part of the cave occupied by 1 cask.

Then, $\frac{13}{x} + \frac{33}{y} = 1$, which represents the whole cave; (1)

and $\frac{5}{x} + \frac{9}{y} = \frac{1}{3};$ (2)

Multiplying (1) by 5, $\frac{65}{x} + \frac{165}{y} = 5;$ (3)

Multiplying (2) by 13, $\frac{65}{x} + \frac{117}{y} = \frac{13}{3}.$ (4)

Subtracting (4) from (3), $\frac{48}{y} = \frac{2}{3};$

$y = 72;$

$x = 24.$

Hence, the cave would contain 24 bales or 72 casks.

18. Let $x =$ value of first horse;

and $y =$ value of second horse.

Then, $y + 15 - x - 50 = 50;$ (1)

and $y + 50 = 1\frac{2}{3}(x + 15);$ (2)

Transposing and uniting (1), $y - x = 85;$ (3)

Transposing and uniting (2), $y - \frac{5x}{3} = -25.$ (4)

Subtracting (4) from (3), $\frac{2x}{3} = 110;$

whence, $x = 165;$

and $y = 250.$

SUPPLEMENT TO SIMPLE EQUATIONS.

Art. 196. (page 145.)

1. Let x = the smaller part;
and nx = the greater part.

Then, $x+nx=a$;

whence, $x=\dfrac{a}{1+n}$,

and $nx=\dfrac{na}{n+1}$.

Hence, we derive a rule:

Divide the sum by $n+1$ for the first part; multiply the first part by n for the second part.

2. Let mx = the first number;
and nx = the second number.

Then, $mx+nx=a$;

whence, $x=\dfrac{a}{m+n}$;

and $mx=\dfrac{ma}{m+n}$;

and $nx=\dfrac{na}{m+n}$.

Hence, we derive the rule:

I. *To find the first number, multiply the sum by m, and divide by $m+n$.*

II. *To find the second number, multiply the sum by n, and divide by $m+n$.*

3. Let mx = the first number;
and nx = the second number

Then, $mx-nx=a$;

whence, $x=\dfrac{a}{m-n}$.

and $mx=\dfrac{ma}{m-n}$;

and $nx=\dfrac{na}{m-n}$.

Hence, we derive the rule:

I. *To find the first number, multiply the difference by m, and divide by $m-n$.*

II. *To find the second number, multiply the difference by n, and divide by $m-n$.*

4. Let $x=$ the first number;
and $y=$ the second number.
Then, $x+y=a$; (1)
and $n(x+y)=m(x-y)$. (2)

Expanding (2), $nx+ny=mx-my$;
Transposing, $mx-nx=my+ny$;
whence, $x=\dfrac{(m+n)y}{m-n}$;
Substituting in (1), $\dfrac{(m+n)y}{m-n}+y=a$;
whence, $y=\dfrac{a(m-n)}{2m}$;
and $x=\dfrac{a(m+n)}{2m}$.

Hence, we derive the following rule:

To find the first number, multiply the sum by $m+n$, and divide by $2m$. To find the second number, multiply the sum by $m-n$, and divide by $2m$.

5. Let $x=$ the first part;
and $y=$ the second part.
Then, $x+y=a$; (1)
and $x+b=my$. (2)

Subtracting (1) from (2), $b-y=my-a$;
whence, $y=\dfrac{a+b}{m+1}$;
and $x=\dfrac{ma-b}{m+1}$.

Hence, we derive the following rule:

To find the second part, add b to the given number, and divide by $m+1$. To find the first part, subtract b from m times the number, and divide by $m+1$.

6. Let $x=$ B's age;
and $mx=$ A's age.

Then, $mx+a=n(x+a)$;
whence, $x=\dfrac{(n-1)a}{m-n}$;
and $mx=\dfrac{am(n-1)}{m-n}$;

Hence, we derive the following rule:

To find the first age, multiply $(n-1)$ by the number of years between the two periods, and divide by the difference of the multiples; the second is m times the first.

7. Let x = number of days second courier travels.

Then, $bx=(n+x)a$;

whence, $x=\dfrac{na}{b-a}$.

Hence, we derive the following rule:

Divide the distance the first is ahead by the difference in the number of miles each travels a day.

8. Let x = number of lbs. of first kind;

and y = number of lbs of second kind.

$$\begin{array}{lll} \text{Then,} & x+y=m; & (1) \\ \text{and} & ax+by=cm. & (2) \\ \text{Multiplying (1) by } a, & ax+ay=am; & (3) \\ \text{Subtracting (2) from (3),} & y=\dfrac{(a-c)m}{a-b}; & \\ \text{and} & x=\dfrac{(c-b)m}{a-b}. & \end{array}$$

Hence, we derive the following rule:

To find the quantity of either kind, multiply the difference between the average price and the price of the other kind by the number of pounds in the mixture, and divide by the difference of the prices of the two kinds.

Art. 200. (page 147.)

Enunciation of problems 5, 6, 7 and 8, that the results may be arithmetically true.

5. What number must be *subtracted* from 18 that the result may be 15?

6. What number must be *added* to 12 that the result may be 15?

7. Required a number such that $\frac{3}{4}$ of it shall exceed $\frac{2}{3}$ of it by 2.

8. A man was born in 1825, and his son in 1855; find how many years before 1870 the father's age is 4 times the son's age.

9. A man labored 10 days, his little son being with him 8 days, and received $18; at another time he labored 14 days, his son being with him 12 days, and received $25; required the wages of the father and the amount paid for the son's board.

Art. 203. (page 151.)

OPERATION.

Let x = the number;
then, $mx - nx = a$;
and $x = \dfrac{a}{m-n}$.

1. The result will be negative when n is greater than m. The result will be indeterminate when the two values are equal and the difference of products is 0, for we shall then have $x = \frac{0}{0}$. The result will be infinite when a has some value and $m = n$, for we shall then have $x = \frac{a}{0} = \infty$.

1st. Suppose $m > n$; then x is positive, and the equation correctly written.

2d. Suppose $m < n$; then x is negative, and mx should be subtracted from nx.

3d. Suppose $m = n$; then $x = \frac{a}{0}$ or ∞, and the equation is impossible. It is evident that, m and n being equal, their products by no finite quantity can differ by a.

4th. Suppose $a = 0$; then $x = \dfrac{0}{m-n} = 0$, for m and n not being equal, their products by zero only will differ by 0.

5th. Suppose $m = n$ and $a = 0$; then $x = \frac{0}{0}$, or anything, for m and n being equal, their products by any number will be equal, or differ by 0.

If we make $m = 4$, $n = 3$, and $a = 6$, then $x = \dfrac{6}{4-3}$, or 6. Suppose $m = 3$ and $n = 5$, $a = 6$, then $x = -3$. This result can be made positive by letting $nx - mx = a$; whence $x = \dfrac{a}{n-m}$, or 3. Suppose $m = 3$ and $n = 3$ and $a = 0$, then $x = \frac{0}{0}$, for $3x - 3x = 0$, an equation that is true for any value of x. Suppose $m = 3$, $n = 3$ and $a = 6$, then $x = \frac{6}{0}$, or ∞, for no value of x will satisfy the equation $3x - 3x = 6$.

OPERATION.

Let x = number of years;
then, $am + x = n(a + x)$;
and $x = \dfrac{a(m-n)}{n-1}$.

2. DISCUSSION. Let x = the number of years between the present and the required time. Then $am + x$ = A's age at the required time, and $a + x$ = B's age. A's age equals n times B's, hence $am + x = an + nx$, from which we find $x = a\dfrac{(m-n)}{n-1}$.

1st. Suppose $m>n$; then x is positive, and the time is future.

2d. Suppose $m<n$; then x is negative, and the time is past.

3d. Suppose $n<1$; then x is negative, and cannot be made positive except by making m also less than 1. In this case A's age is a fractional part of B's. If $m>1$ and $n<1$, the problem is absurd, for in one case A's age is a multiple of B's, and in the other a part of it; in other words, he is at one time older and at another time younger than B.

3. DISCUSSION. Let $x=$ the time past m o'clock, or the distance the minute-hand goes; then $\frac{x}{12}=$ the distance the hour-hand goes. To overtake the hour-hand the minute-hand must gain $5m$ spaces; but since they are a minute-spaces apart, the minute-hand gains either $5m-a$, or $5m+a$ spaces. Hence, $x=\frac{(5m\mp a)12}{11}$.

OPERATION.

Let $x=$ the time past m o'clock;

$x-\frac{x}{12}=5m-a$, or $5m+a$;

$$x=\frac{12(5m\mp a)}{11}.$$

1st. Suppose $5m>a$; then x is positive, and the minute-hand a spaces on one side or the other of the hour-hand, as we use the different signs.

2d. Suppose $5m<a$; then, if we take the negative value of a, x is negative, and the minute-hand will be $\frac{12(5m-a)}{11}$ minutes before m o'clock. If we take the positive value of a, x is positive, and the time will be $\frac{12(5m+a)}{11}$ minutes after m o'clock.

3d. Suppose $5m=a$; then, $x=12\left(\frac{0}{11}\right)=0$, and the time is exactly m o'clock.

Art. 205. (page 154.)

2. Let $x=$ the number; then, $\frac{x}{4}+\frac{x}{6}=\frac{11x}{20}-\frac{2x}{15}$.

Clearing of fractions, $15x+10x=33x-8x$; uniting, $0\times x=0$; whence, $x=\frac{0}{0}$. Indeterminate.

This result indicates that any number will fulfill the conditions. By adding the fractions, we have $\frac{1}{4}+\frac{1}{6}=\frac{5}{12}$; by subtracting, we have $\frac{11}{20}-\frac{2}{15}=\frac{5}{12}$; it is thus evident that any number will satisfy the conditions, since $\frac{5}{12}$ of *a number* will always be equal to $\frac{5}{12}$ of the *same number*.

3. Let x = the number. Then, $\frac{5x}{6}-4=\frac{x}{2}+\frac{x}{3}-3$; $0\times x=1$; $x=\frac{1}{0}=\infty$.

This result indicates that no finite number will fulfill the conditions. If we add the fractions in the second member, we have $\frac{5}{6}$ of a number -4 $=\frac{5}{6}$ of the same number -3, which is evidently impossible.

4. Let x = number of days A labored, and y = number of days B labored. Then, $2x+3y=20$.

Here we have two unknown quantities and only one equation. We may give x any value we please, and y will have a corresponding value; therefore the problem may have an indefinite number of answers, and is indeterminate.

5. Let x = the number of years. $60+x=30+x$; $0.x=30$; $x=\frac{30}{0}=\infty$.

The result indicates that the problem is absurd. This is also evident, since, though the *ratio* of A's age to his son's age is *diminishing*, the actual *difference* remains the *same*.

6. Let $\frac{x}{y}$ = the fraction.

Then, $\frac{x-2}{y}=\frac{2}{3}$; (1)

and $\frac{x}{y+3}=\frac{2}{3}$. (2)

Clearing (1) of fractions, $3x-6=2y$; (3)

Clearing (2) of fractions, $3x=2y+6$. (4)

Here the equations are alike, hence we really have but one equation for two unknown quantities, and the problem is indeterminate. By assigning values to x corresponding values of y may be obtained. When x is 4, y will be 3; when x is 6, y will be 6; when x is 8, y will be 9; when x is 10, y will be 12, etc. Hence, $\frac{4}{3}$, $\frac{6}{6}$, $\frac{8}{9}$, $\frac{10}{12}$, etc., will satisfy the conditions of the problem.

7. Let x = the number.

Then, $\frac{4x-12}{x-3}=\frac{4x+9}{x+3}$;

Dividing both terms of first member by $x-3$, $4=\frac{4x+9}{x+3}$;

Clearing of fractions, $4x+12=4x+9$;

whence, $0.x=3$;

$$x=\frac{3}{0}=\infty.$$

This result indicates that the problem is impossible. If we solve it by another method, we will obtain an *apparent root,* which, however, will not satisfy the equation. See the following solution.

If we clear equation (1) of fractions, transpose and unite, we have $x=3$. This apparent value of x will not verify in equation (1), but verifies in (2). This is accounted for by observing that $x-3$ *is introduced into the equation as a new factor when we clear of fractions,* and therefore 3 is a true value of x for equation (2), but not for equation (1).

OPERATION.

$$\frac{4x-12}{x-3}=\frac{4x+9}{x+3} \qquad (1)$$

$$4x^2-36=4x^2-3x-27 \qquad (2)$$

$$x=3 \qquad (3)$$

INVOLUTION.

Art. 211. (page 158.)

17. $\dfrac{a^{2n}(x-2)}{a^n(x-3)}=\dfrac{a^n(x-2)}{x-3}$. Squaring fraction, we have $\dfrac{a^{2n}(x-2)^2}{(x-3)^2}$.

18. $\dfrac{a^3(a^2-4)}{a^3-5a^2+6a}$. Dividing both terms by $a(a-2)$, the fraction becomes $\dfrac{a^2(a+2)}{a-3}$, the cube of which equals $\dfrac{a^6(a+2)^3}{(a-3)^3}$.

Art. 221. (page 166.)

7. Let $3z=m$ and $\dfrac{1}{z}=n$; then, $(m-n)^4$ will equal $\left(3z-\dfrac{1}{z}\right)^4$.

$(m-n)^4=m^4-4m^3n+6m^2n^2-4mn^3+n^4$.

Substituting, $(3z)^4-4(3z)^3\dfrac{1}{z}+6(3z)^2\left(\dfrac{1}{z}\right)^2-4(3z)\left(\dfrac{1}{z}\right)^3+\left(\dfrac{1}{z}\right)^4$.

Reducing, $81z^4-108z^2+54-\dfrac{12}{z^2}+\dfrac{1}{z^4}$.

8. Let $\frac{1}{2}a=m$ and $\frac{2}{3}c=n$; then, $(m-n)^5$ will equal $(\frac{1}{2}a-\frac{2}{3}c)^5$.

$(m-n)^5=m^5-5m^4n+10m^3n^2-10m^2n^3+5mn^4-m^5$.

Substituting, $\left(\dfrac{a}{2}\right)^5-5\left(\dfrac{a}{2}\right)^4\dfrac{2c}{3}+10\left(\dfrac{a}{2}\right)^3\left(\dfrac{2c}{3}\right)^2-10\left(\dfrac{a}{2}\right)^2\left(\dfrac{2c}{3}\right)^3+5\left(\dfrac{a}{2}\right)\left(\dfrac{2c}{3}\right)^4-\left(\dfrac{2c}{3}\right)^5$.

Reducing, $\dfrac{a^5}{32}-\dfrac{5a^4c}{24}+\dfrac{5a^3c^2}{9}-\dfrac{20a^2c^3}{27}+\dfrac{40ac^4}{81}-\dfrac{32c^5}{243}$.

9. Let $n^2=a$ and $n^{-2}=b$; then, $(a-b)^6$ will equal $(n^2-n^{-2})^6$.
$(a-b)^6=a^6-6a^5b+15a^4b^2-20a^3b^3+15a^2b^4-6ab^5+b^6$.
Substituting, $(n^2)^6-6(n^2)^5n^{-2}+15(n^2)^4(n^{-2})^2-20(n^2)^3(n^{-2})^3$
$+15(n^2)^2(n^{-2})^4-6n^2(n^{-2})^5+(n^{-2})^6$.
Reducing, $n^{12}-6n^8+15n^4-20+15n^{-4}-6n^{-8}+n^{-12}$.

10. $(1+\frac{3}{2}x)^5=1+5(\frac{3}{2}x)+10(\frac{3}{2}x)^2+10(\frac{3}{2}x)^3+5(\frac{3}{2}x)^4+(\frac{3}{2}x)^5$.
Reducing, $1+\dfrac{15x}{2}+\dfrac{45x^2}{2}+\dfrac{135x^3}{4}+\dfrac{405x^4}{16}+\dfrac{243x^5}{32}$.

11. Let $2a^2x=m$ and $4az^3=n$; then, $(m-n)^6=(2a^2x-4az^3)^6$.
$(m-n)^6=m^6-6m^5n+15m^4n^2-20m^3n^3+15m^2n^4-6mn^5+n^6$.
Substituting, $(2a^2x)^6-6(2a^2x)^5(4az^3)+15(2a^2x)^4(4az^3)^2-20(2a^2x)^3$
$(4az^3)^3+15(2a^2x)^2(4az^3)^4-6(2a^2x)(4az^3)^5+(4az^3)^6$.
Reducing, $64a^{12}x^6-768a^{11}x^5z^3+3840a^{10}x^4z^6-10240a^9x^3z^9+15360a^8x^2z^{12}$
$-12288a^7xz^{15}+4096a^6z^{18}$.

Art. 222. (page 167.)

3. Let $a=x$ and $b-c=y$; then, $(a+b-c)^3=(x+y)^3$.
$(x+y)^3=x^3+3x^2y+3xy^2+y^3$.
Substituting, $a^3+3a^2(b-c)+3a(b-c)^2+(b-c)^3$.
Expanding, $a^3+3a^2b-3a^2c+3ab^2-6abc+3ac^2+b^3-3b^2c+3bc^2-c^3$.

4. Let $a+b=x$ and $c+d=y$; then, $(a+b+c+d)^3=(x+y)^3$.
$(x+y)^3=x^3+3x^2y+3xy^2+y^3$.
Substituting, $(a+b)^3+3(a+b)^2(c+d)+3(a+b)(c+d)^2+(c+d)^3$.
Expanding, $a^3+3a^2b+3ab^2+b^3+3a^2c+6abc+3b^2c+3a^2d+6abd$
$+3b^2d+3ac^2+6acd+3ad^2+3bc^2+6bcd+3bd^2+c^3+3c^2d$
$+3cd^2+d^3$.

MISCELLANEOUS EXAMPLES.

1. Let x and y be the numbers; then $x^2+2xy+y^2=$ the square of their sum, and $x^2-2xy+y^2=$ the square of their difference. Then $x^2+2xy+y^2-(x^2-2xy+y^2)=4xy$.

2. Let x and y be the numbers; then $x^2+2xy+y^2=$ the square of their sum, and $x^2-y^2=$ the product of their sum and difference. Then $x^2+2xy+y^2-(x^2-y^2)=2xy+2y^2$, or, *the square of their sum exceeds the product of their sum and difference by twice their product, plus twice the square of the lesser number.*

3. Let $x=$ the number; then $\frac{x^2}{2}=$ one-half of its square, and $\left(\frac{x}{2}\right)^2$ or $\frac{x^2}{4}=$ the square of one-half of the number. Then $\frac{x^2}{2}-\frac{x^2}{4}=\frac{x^2}{4}$, or the square of one-half of the number.

4. Let x and $x+1$ be the numbers. Then $x^2+2x+1-x^2=2x+1$.

5. Let x and $x+1$ be the numbers. Then $(x+1)^2-x^2=2x+1$; and $x+1+x=2x+1$.

6 Let $x-1$, x, and $x+1$ be the numbers. Then $x^2=$ the square of the second, and $(x-1)(x+1)=x^2-1$, the product of first and third. Then $x^2-(x^2-1)=1$.

7. Let $x-1$, x, and $x+1$ be the numbers. Then the sum of the cubes equals $x^3-3x^2+3x-1+x^3+x^3+3x^2+3x+1=3x^3+6x$; and the sum of the numbers equals $x-1+x+x+1=3x$; $(3x^3+6x)\div 3x=x^2+2$; hence the sum of the cubes is divisible by the sum of the numbers.

Art. 234. (page 172.)

15. Arranging the terms inversely according to the powers of n, we proceed as in the rule.

$$
\begin{array}{r|l}
 & 1-4n+10n^2-20n^3+25n^4-24n^5+16n^6 \quad \underline{|1-2n+3n^2-4n^3} \\
 & 1 \\
2-2n & -4n+10n^2 \\
 & -4n+\ \ 4n^2 \\
2-4n+\ 3n^2 & 6n^2-20n^3+25n^4 \\
 & 6n^2-12n^3+\ 9n^4 \\
2-4n+6n^2-4n^3 & -\ 8n^3+16n^4-24n^5+16n^6 \\
 & -\ 8n^3+16n^4-24n^5+16n^6
\end{array}
$$

16. Collecting and arranging the terms according to the powers of a, we proceed as in the rule.

$$
\begin{array}{r|l}
 & a^6-4a^5x+8a^4x^2-10a^3x^3+8a^2x^4-4ax^5+x^6 \quad \underline{|a^3-2a^2x+2ax^2-x^3} \\
 & a^6 \\
2a^3-2a^2x & -4a^5x+8a^4x^2 \\
 & -4a^5x+4a^4x^2 \\
2a^3-4a^2x+2ax^2 & 4a^4x^2-10a^3x^3+8a^2x^4 \\
 & 4a^4x^2-\ 8a^3x^3+4a^2x^4 \\
2a^3-4a^2x+4ax^2-x^3 & -2a^3x^3+4a^2x^4-4ax^5+x^6 \\
 & -2a^3x^3+4a^2x^4-4ax^5+x^6
\end{array}
$$

Art. 237. (page 177.)

5.

$$\begin{array}{r|l}
 & a^3+3a^2b+3ab^2+b^3+3a^2c+6abc+3b^2c+3ac^2+3bc^2+c^3 \quad \underline{a+b+c} \\
 & a^3 \\ \hline
3a^2 & 3a^2b+3ab^2+b^3 \\
3a^2+3ab+b^2 & 3a^2b+3ab^2+b^3 \\ \hline
3a^2+6ab+3b^2 & 3a^2c+6abc+3b^2c+3ac^2+3bc^2+c^3 \\
3a^2+6ab+3b^2+3ac+3bc+c^2 & 3a^2c+6abc+3b^2c+3ac^2+3bc^2+c^3
\end{array}$$

6.

$$\begin{array}{r|l}
 & a^6-3a^5+5a^3-3a-1 \quad \underline{a^2-a-1} \\
 & a^6 \\ \hline
3a^4 & -3a^5+5a^3-3a-1 \\
3a^4-3a^3+a^2 & -3a^5+3a^4-a^3 \\ \hline
3a^4-6a^3+3a^2 & -3a^4+6a^3-3a-1 \\
3a^4-6a^3+3a^2-3a^2+3a+1 & -3a^4+6a^3-3a-1
\end{array}$$

7.

$$\begin{array}{r|l}
 & a^6-6a^5+15a^4-20a^3+15a^2-6a+1 \quad \underline{a^2-2a+1} \\
 & a^6 \\ \hline
3a^4 & -6a^5+15a^4-20a^3 \\
3a^4-6a^3+4a^2 & -6a^5+12a^4-8a^3 \\ \hline
3a^4-12a^3+12a^2 & 3a^4-12a^3+15a^2-6a+1 \\
3a^4-12a^3+15a^2-6a+1 & 3a^4-12a^3+15a^2-6a+1
\end{array}$$

8.

$$\begin{array}{r|l}
 & m^6+6m^5-40m^3+96m-64 \quad \underline{m^2+2m-4} \\
 & m^6 \\ \hline
3m^4 & 6m^5-40m^3+96m \\
3m^4+6m^3+4m^2 & 6m^5+12m^4+8m^3 \\ \hline
3m^4+12m^3+12m^2 & -12m^4-48m^3+96m-64 \\
3m^4+12m^3+12m^2-12m^2+24m+16 & -12m^4-48m^3+96m-64
\end{array}$$

9.

$$\begin{array}{r|l}
 & x^6-3ax^5+5a^3x^3-3a^5x-a^6 \quad \underline{x^2-ax-a^2} \\
 & x^6 \\ \hline
3x^4 & -3ax^5+5a^3x^3 \\
3x^4-3ax^3+a^2x^2 & -3ax^5+3a^2x^4-a^3x^3 \\ \hline
3x^4-6ax^3+3a^2x^2 & -3a^2x^4+6a^3x^3-3a^5x-a^6 \\
3x^4-6ax^3+3a^2x^2-3a^2x^2+3a^3x+a^4 & -3a^2x^4+6a^3x^3-3a^5x-a^6
\end{array}$$

Art. 238. (page 180.)

A few of these problems will be solved by the *second method.*

	1st Col.	2d Col.		
10.			1·879·080·904	1234
	1		1	
	2	3 .. t.d.	879	
	32	64		
	4	364 C.D.	728	
	363	4	151080	
	6	432 .. t.d.		
	3694	1089		
		44289 C.D.	132867	
		9	18213904	
		45387 .. t. d.		
		14776		
		4553476 C.D.	18213904	

	1st Col.	2d Col.		
11.			41·063·625	34.5
	3		27	
	6	27 .. t.d.	14063	
	94	376		
	8	3076 C.D.	12304	
	1025	16	1759625	
		3468 .. t.d.		
		5125		
		351925 C.D.	1759625	

	1st Col.	2d Col.		
12.			130·323·843	5.07
	5		125	
	100	7500 .. t.d.	5323843	
	1507	10549		
		760549 C.D.	5323843	

	1st. Col.	2d. Col.		
13.			95·256·152·263	4567
	4		64	
	8	48 .. t.d.	31256	
	125	625		
	10	5425 C.D.	27125	
	1356	25	4131152	
	12	6075 .. t.d.		
	13687	8136		
		615636 C.D.	3693816	
		36	437336263	
		623808 .. t.d.		
		95809		
		62476609 C.D.	437336263	

16.

1st Col.	2d Col.	7·000000000	1.9129+
1		1	
2	3 .. t.d.	6000	
39	351		
18	651 C.D.	5859	
571	81	141000	
2	1083 .. t.d.		
5732	571		
4	108871 C.D.	108871	
57369	1	32129000	
	109443 .. t d.		
	11464		
	10955764 C.D.	21911528	
	4	10217472000	
	10967232 .. t.d.		
	516321		
	1097239521 C.D.	9875155689	
		342316311	

Art. 248. (page 184.)

9. $2\sqrt{9(a^3-a^2c)}=2\sqrt{\{9a^2(a-c)\}}=6a\sqrt{(a-c)}$.

10. $a\sqrt[3]{(x^3-x^3z)}=a\sqrt[3]{x^3(1-z)}=ax\sqrt[3]{(1-z)}$.

11. $2(ax^2-bx^3)^{\frac{1}{2}}=2\{x^2(a-bx)\}^{\frac{1}{2}}=2x(a-bx)^{\frac{1}{2}}$.

12. $5a\sqrt[3]{(54a^2b^3c^4)}=5a\sqrt[3]{\{27b^3c^3(2a^2c)\}}=15abc\sqrt[3]{2a^2c}$.

13. $(75a^3x^5y)^{\frac{1}{2}}=\{25a^2x^4(3axy)\}^{\frac{1}{2}}=5ax^2(3axy)^{\frac{1}{2}}$.

14. $(24a^4b^5c^2)^{\frac{1}{3}}=\{8a^3b^3(3ab^2c^2)\}^{\frac{1}{3}}=2ab(3ab^2c^2)^{\frac{1}{3}}$.

15. $\sqrt[4]{\{(162a^4(b^5-b^4c)^{\frac{1}{4}}\}}=\sqrt[4]{\{81\times 2a^4b(b-c)^{\frac{1}{4}}\}}=3a\sqrt[4]{\{2b(b-c)^{\frac{1}{4}}\}}$,

16. $(a+b)(a^3-2a^2b+ab^2)^{\frac{1}{2}}=(a+b)\{(a-b)^2a\}^{\frac{1}{2}}=(a+b)(a-b)\sqrt{a}$
$=(a^2-b^2)\sqrt{a}$.

17. $(m-n)(2am^2+4amn+2an^2)^{\frac{1}{2}}=(m-n)\{2a(m+n)^2\}^{\frac{1}{2}}=(m-n)$
$(m+n)\sqrt{2a}=(m^2-n^2)\sqrt{2a}$.

18. $2a\sqrt{(8x^3y+16x^2y^2+8xy^3)}=2a\sqrt{\{8xy(x+y)^2\}}=4a(x+y)\sqrt{(2xy)}$.

Art. 249. (page 185.)

5. $2\sqrt{\frac{4a^3}{5}}=2\sqrt{\frac{4a^3\times 5}{5\times 5}}=2\sqrt{\frac{20a^3}{25}}=\frac{4a}{5}\sqrt{5a}.$

6. $2\sqrt{\frac{5}{8}}=2\sqrt{\frac{5\times 2}{8\times 2}}=2\sqrt{\frac{10}{16}}=\frac{1}{2}\sqrt{10}.$

7. $2\sqrt[3]{\frac{4a^4}{9}}=2\sqrt[3]{\frac{4a^4\times 3}{9\times 3}}=2\sqrt[3]{\frac{12a^4}{27}}=\frac{2a}{3}\sqrt[3]{12a}.$

8. $4b\sqrt[3]{\frac{3a^4}{4b^2}}=4b\sqrt[3]{\frac{6a^4b}{8b^3}}=\frac{4ab}{2b}\sqrt[3]{6ab}=2a\sqrt[3]{6ab}.$

9. $6z^2\left(\frac{8x^5y^2}{27z^3}\right)^{\frac{1}{4}}=6z^2\left(\frac{24x^5y^2z}{81z^4}\right)^{\frac{1}{4}}=\frac{6xz^2}{3z}(24xy^2z)^{\frac{1}{4}}=2xz(24xy^2z)^{\frac{1}{4}}.$

Art. 250. (page 186.)

9. $\frac{2}{3}\sqrt{3ax}=\sqrt{\frac{4}{9}(3ax)}=\sqrt{\frac{4ax}{3}}.$

10. $6a\sqrt{cx}=2\times 3a\sqrt{cx}=2\sqrt{9a^2cx}.$

11. $\frac{2a}{b}\sqrt{2ab^3}=2\times\frac{a}{b}\sqrt{2ab^3}=2\sqrt{\left(\frac{a^2}{b^2}\times 2ab^3\right)}=2\sqrt{(2a^3b)}.$

12. $\frac{3c}{4}\sqrt[3]{\frac{4ax}{9c^2}}=\sqrt[3]{\left(\frac{27c^3}{64}\times\frac{4ax}{9c^2}\right)}=\sqrt[3]{\frac{3acx}{16}}.$

Art. 251. (page 187.)

6. $\sqrt{3}=\sqrt[12]{3^6}=\sqrt[12]{729}$; $\sqrt[3]{4}=\sqrt[12]{4^4}=\sqrt[12]{256}$; $\sqrt[4]{5}=\sqrt[12]{5^3}=\sqrt[12]{125}.$

7. $2\sqrt{6}=2\sqrt[6]{6^3}=2\sqrt[6]{216}$; $3\sqrt[3]{9}=3\sqrt[6]{9^2}=3\sqrt[6]{81}$; $5\sqrt[6]{8}.$

8. $\sqrt{2a}=\sqrt[12]{(2a)^6}=\sqrt[12]{64a^6}$; $\sqrt[4]{3a^3}=\sqrt[12]{(3a^3)^3}=\sqrt[12]{27a^9}$; $\sqrt[6]{4a^3}$ $=\sqrt[12]{(4a^3)^2}=\sqrt[12]{16a^6}.$

Art. 252. (page 188.)

10. $2\sqrt[3]{16a}=4\sqrt[3]{2a}$
$3\sqrt[3]{54a}=9\sqrt[3]{2a}$
Sum $=13\sqrt[3]{2a}$

11. $\sqrt{50}=5\sqrt{2}$
$\sqrt{72}=6\sqrt{2}$
$\sqrt{128}=8\sqrt{2}$
Sum $=19\sqrt{2}$

12. $\sqrt{28a^2c^3}=2ac\sqrt{7c}$

$c\sqrt{112a^2c}=4ac\sqrt{7c}$

$\text{Sum}=6ac\sqrt{7c}$

13. $\sqrt{2}=\sqrt{2}$

$2\sqrt{\tfrac{1}{2}}=2\sqrt{\tfrac{2}{4}}=\sqrt{2}$

$\text{Sum}=2\sqrt{2}$

14. $2\sqrt{3}=2\sqrt{3}$

$3\sqrt{\tfrac{1}{3}}=3\sqrt{\tfrac{3}{9}}=\sqrt{3}$

$\text{Sum}=3\sqrt{3}$

15. $4\sqrt[3]{\tfrac{1}{4}}=4\sqrt[3]{\tfrac{2}{8}}=2\sqrt[3]{2}$

$6\sqrt[3]{\tfrac{1}{32}}=6\sqrt[3]{\tfrac{2}{64}}=\tfrac{3}{2}\sqrt[3]{2}$

$\text{Sum}=\tfrac{7}{2}\sqrt[3]{2}$

16. $\sqrt{(2a^2x)}=a\sqrt{2x}$

$\sqrt{(2b^2x)}=b\sqrt{2x}$

$\text{Sum}=(a+b)\sqrt{2x}$

17. $\sqrt{a^2m}=a\sqrt{m}$

$\sqrt{a^2n}=a\sqrt{n}$

$\text{Sum}=a(\sqrt{m}+\sqrt{n})$

18. $\sqrt{a^4c}=a^2\sqrt{c}$

$2\sqrt{a^2b^2c}=2ab\sqrt{c}$

$\sqrt{b^4c}=b^2\sqrt{c}$

$\text{Sum}=(a+b)^2\sqrt{c}$

19. $2x\sqrt{50a^3c}=10ax\sqrt{2ac}$

$3\sqrt[3]{24a^4x^3}=6ax\sqrt[3]{3a}$

$\tfrac{1}{3}a\sqrt{72acx^2}=2ax\sqrt{2ac}$

$2x\sqrt[3]{81a^4}=6ax\sqrt[3]{3a}$

$\text{Sum}=12ax\sqrt{2ac}+12ax\sqrt[3]{3a}$

$=12ax(\sqrt{2ac}+\sqrt[3]{3a})$

Art. 253. (page 189.)

3. $\sqrt{(49ax^3)}=7x\sqrt{ax}$

$\sqrt{(25ax^3)}=5x\sqrt{ax}$

$\text{Difference}=2x\sqrt{ax}$

4. $3\sqrt{12a^3}=6a\sqrt{3a}$

$a\sqrt{27a}=3a\sqrt{3a}$

$\text{Difference}=3a\sqrt{3a}$

5. $\sqrt[3]{(125a^2)}=5\sqrt[3]{a^2}$

$\sqrt[3]{8a^2}=2\sqrt[3]{a^2}$

$\text{Difference}=3\sqrt[3]{a^2}$

6. $2\sqrt{a^2c}=2a\sqrt{c}$

$a\sqrt{c^3}=ac\sqrt{c}$

$\text{Difference}=(2a-ac)\sqrt{c}$

7. $\sqrt{12a}=2\sqrt{3a}$

$2\sqrt{\dfrac{3a}{4}}=\sqrt{3a}$

$\text{Difference}=\sqrt{3a}$

8. $\sqrt[3]{250a^4x}=5a\sqrt[3]{2ax}$

$\sqrt[3]{54a^4x}=3a\sqrt[3]{2ax}$

$\text{Difference}=2a\sqrt[3]{2ax}$

9. $3\sqrt{\tfrac{3}{4}}=\tfrac{3}{2}\sqrt{3}$

$2\sqrt{\tfrac{1}{3}}=\tfrac{2}{3}\sqrt{3}$

$\text{Difference}=\tfrac{5}{6}\sqrt{3}$

10. $4\sqrt[4]{32}=4\sqrt[4]{16\times 2}=8\sqrt[4]{2}$

$4\sqrt[4]{\tfrac{1}{8}}=4\sqrt[4]{\tfrac{2}{16}}=2\sqrt[4]{2}$

$\text{Difference}=6\sqrt[4]{2}$

11. $$\begin{aligned}\sqrt{(a^3+a^2x)} &= a\sqrt{(a+x)}\\ \sqrt{(9ab^2+9b^2x)} &= 3b\sqrt{(a+x)}\\ \hline \text{Difference} &= (a-3b)\sqrt{(a+x)}\end{aligned}$$

12. $$\begin{aligned}\sqrt{(2a^3+4a^2b+2ab^2)} &= (a+b)\sqrt{2a}\\ \sqrt{(2a^3-4a^2b+2ab^2)} &= (a-b)\sqrt{2a}\\ \hline \text{Difference} &= 2b\sqrt{2a}\end{aligned}$$

13. $$\begin{aligned}7b\sqrt{a^3x} &= 7ab\sqrt{ax}\\ -a\sqrt{9ab^2x} &= -3ab\sqrt{ax}\\ +5\sqrt{a^3c^2x} &= 5ac\sqrt{ax}\\ -3c\sqrt{9a^3x} &= -9ac\sqrt{ax}\\ \hline \text{Sum} &= 4a(b-c)\sqrt{ax}\end{aligned}$$

Art. 256. (page 191.)

7. $$\begin{array}{l} 2\sqrt{27}\\ 3\sqrt{3}\\ \hline 6\sqrt{81}=54\end{array}$$

8. $$\begin{array}{l} 5\sqrt[3]{4a}\\ 3\sqrt[3]{2a}\\ \hline 15\sqrt[3]{8a^2}=30\sqrt[3]{a^2}\end{array}$$

9. $$\begin{array}{l} 2\sqrt{\frac{1}{2}}\\ 2\sqrt{\frac{5}{8}}\\ \hline 4\sqrt{\frac{5}{16}}=\sqrt{5}\end{array}$$

10. $$\begin{array}{l} 3\sqrt{\frac{a}{3}}\\ 2\sqrt{\frac{a}{6}}\\ \hline 6\sqrt{\frac{a^2}{18}}=6\sqrt{\frac{2a^2}{36}}=a\sqrt{2}\end{array}$$

11. $$\begin{array}{l} 3\sqrt[n]{(a^2x)}\\ \sqrt[n]{(ax^3)}\\ \hline 3\sqrt[n]{(a^3x^4)}\end{array}$$

12. $$\begin{array}{l} \sqrt[n]{(a^{n-1}c^{n+1})}\\ \sqrt[n]{(ac^n)}\\ \hline \sqrt[n]{(a^nc^{2n+1})}=ac^2\sqrt[n]{c}\end{array}$$

13. $$\begin{array}{l} a\sqrt[n]{(b^{n+1}c^{n-1})}\\ \sqrt[n]{(a^{n+1}c^3)}\\ \hline a\sqrt[n]{(a^{n+1}b^{n+1}c^{n+2})}=a^2bc\sqrt[n]{(abc^2)}\end{array}$$

14. $$\begin{array}{l} \sqrt{a}+\sqrt{b}\\ \sqrt{a}-\sqrt{b}\\ \hline a+\sqrt{ab}\\ \quad -\sqrt{ab}-b\\ \hline a \qquad\quad -b\end{array}$$

18. $$(a+b)^{\frac{n}{2}}\times(a+b)^{\frac{n}{2}}=(a+b)^{\frac{2n}{2}}=(a+b)^n$$

Art. 257. (page 191.)

3. $$\begin{aligned}\sqrt{a} &= a^{\frac{1}{2}}\\ \sqrt[3]{a} &= a^{\frac{1}{3}}\\ \hline \text{Product } &= a^{\frac{5}{6}}.\end{aligned}$$

4. $$\begin{aligned} &3a^{\frac{1}{3}}\\ 4(ab)^{\frac{2}{3}} &= 4a^{\frac{2}{3}}b^{\frac{2}{3}}\\ \hline \text{Product } &= 12ab^{\frac{2}{3}}.\end{aligned}$$

5. $$a\sqrt[n]{b} = a\sqrt[mn]{b^m}$$
$$b\sqrt[m]{c} = b\sqrt[mn]{c^n}$$
$$\text{Product} = ab\sqrt[mn]{b^m c^n}.$$

6. $$3\sqrt[m]{a} = 3a^{\frac{1}{m}}$$
$$4\sqrt[n]{a} = 4a^{\frac{1}{n}}$$
$$\text{Product} = 12a^{\frac{m+n}{mn}}.$$

7. $$a\sqrt[n]{c^{2n}} = ac^2$$
$$a\sqrt[m]{c^{3m}x} = ac^3\sqrt[m]{x}$$
$$\text{Product} = a^2c^5\sqrt[m]{x}.$$

8. $$\sqrt{(a+c)} = \sqrt[6]{(a+c)^3}$$
$$\sqrt[3]{(a+c)} = \sqrt[6]{(a+c)^2}$$
$$\text{Product} = \sqrt[6]{(a+c)^5}.$$

9. $$2\sqrt[n]{(a-c)} = 2\sqrt[2n]{(a-c)^2}$$
$$3\sqrt{a} = 3\sqrt[2n]{a^n}$$
$$\text{Product} = 6\sqrt[2n]{\{a^n(a-c)^2\}}.$$

10. $$\sqrt{(m+n)} = \sqrt[6]{(m+n)^3}$$
$$\sqrt[3]{(m-n)} = \sqrt[6]{(m-n)^2}$$
$$\text{Product} = \sqrt[6]{(m+n)(m^2-n^2)^2}.$$

Art. 260. (page 192.)

4. $$\frac{5\sqrt{27ac}}{3\sqrt{3a}} = \tfrac{5}{3}\sqrt{9c} = 5\sqrt{c}.$$

5. $$\frac{6\sqrt{54a}}{3\sqrt{27}} = 2\sqrt{2a}.$$

6. $$\frac{3\sqrt{72ab}}{2\sqrt{6b}} = \tfrac{3}{2}\sqrt{12a}$$
$$= 3\sqrt{3a}.$$

7. $$\frac{6\sqrt{20a}}{2\sqrt{30a}} = 3\sqrt{\tfrac{2}{3}} = \sqrt{6}.$$

8. $$\frac{2\sqrt{a^{\frac{7}{3}}}}{\sqrt{2a^{\frac{1}{3}}}} = 2\sqrt{\tfrac{1}{2}a^{\frac{6}{3}}} = 2a\sqrt{\tfrac{1}{2}}$$
$$= a\sqrt{2}.$$

9. $$\frac{15(a^3b^5)^{\frac{1}{2}}}{3(ab^2)^{\frac{1}{2}}} = 5(a^2b^3)^{\frac{1}{2}} = 5ab\sqrt{b}.$$

10. $$\frac{5\sqrt[3]{16a^2x^4}}{2\sqrt[3]{2ax}} = \tfrac{5}{2}\sqrt[3]{8ax^3} = 5x\sqrt[3]{a}.$$

11. $$\frac{\frac{3}{4}\sqrt{\frac{1}{3a}}}{\frac{1}{2}\sqrt{\frac{3a}{5}}} = \tfrac{3}{2}\sqrt{\frac{5}{9a^2}} = \frac{1}{2a}\sqrt{5}.$$

12. $$\frac{(1-a^2)^{\frac{1}{2}}}{(1-a)^{\frac{1}{2}}} = (1+a)^{\frac{1}{2}}.$$

Art. 261. (page 193.)

3. $$\frac{6\sqrt[3]{(a^2x^4)}}{2\sqrt{(ax)}} = \frac{6\sqrt[6]{(a^4x^8)}}{2\sqrt[6]{(a^3x^3)}} = 3\sqrt[6]{(ax^5)}$$

4. $$\frac{2\sqrt{ab}}{2\sqrt[4]{ab}} = \frac{2\sqrt[4]{(a^2b^2)}}{2\sqrt[4]{ab}} = \sqrt[4]{ab}.$$

5. $$\frac{8\sqrt{ax}}{4\sqrt{ax^2}} = 2\sqrt{\frac{1}{x}} = 2\sqrt{x^{-1}}.$$

6. $$\frac{6\sqrt{3}}{3\sqrt[3]{3}} = \frac{6\sqrt[6]{27}}{3\sqrt[6]{9}} = 2\sqrt[6]{3}.$$

7. $$\frac{12}{\sqrt{3}} = \frac{\sqrt{144}}{\sqrt{3}} = \sqrt{48} = 4\sqrt{3}.$$

8. $$\frac{4\sqrt[3]{cz}}{6\sqrt{ac}} = \frac{4\sqrt[6]{c^2z^2}}{6\sqrt[6]{a^3c^3}} = \tfrac{2}{3}\sqrt[6]{\frac{z^2}{a^3c}}.$$

9. $\dfrac{a\sqrt[n]{x}}{c\sqrt[m]{x}}=\dfrac{a\sqrt[mn]{x^m}}{c\sqrt[mn]{x^n}}=\dfrac{a}{c}\sqrt[mn]{x^{m-n}}.$

10. $\dfrac{\frac{4}{5}\sqrt{\frac{a}{c}}}{\frac{2}{3}\sqrt[3]{\frac{a}{c}}}=\frac{6}{5}\dfrac{\sqrt[6]{\frac{a^3}{c^3}}}{\sqrt[6]{\frac{a^2}{c^2}}}=\frac{6}{5}\sqrt[6]{\frac{a}{c}}.$

Art. 262. (page 194.)

6. $\left(4\sqrt[4]{\frac{ax^3}{4}}\right)^3=64\sqrt[4]{\frac{a^3x^9}{64}}=64\sqrt[4]{\frac{4a^3x^9}{256}}=16x^2\sqrt[4]{4a^3x}.$

7. $\left(3\sqrt{\frac{a}{3}}\right)^4=81\sqrt{\frac{a^4}{81}}=9a^2.$

8. $\left(2a\sqrt[3]{\frac{x}{a}}\right)^4=16a^4\sqrt[3]{\frac{x^4}{a^4}}=16a^4\sqrt[3]{\frac{a^2x^4}{a^6}}=16a^2x\sqrt[3]{a^2x}.$

9. $(a\sqrt[3]{x})^n=a^n\sqrt[3]{x^n}.$

10. $(\sqrt[3]{2a^2}\times\sqrt[6]{ax^3})^3=2a^2\sqrt{ax^3}=2a^2x\sqrt{ax}.$

11. $(\sqrt{a}-\sqrt{x})^2=a-2(\sqrt{a})(\sqrt{x})+x=a-2\sqrt{ax}+x.$

12. $(\sqrt{2}+a\sqrt{2})^2=2+2(\sqrt{2})(a\sqrt{2})+2a^2=2+4a+2a^2.$

Art. 263. (page 195.)

5. $(2\sqrt{ax})^{\frac{1}{3}}=(\sqrt{4ax})^{\frac{1}{3}}=\sqrt[6]{4ax}.$

6. $(2a\sqrt{2a})^{\frac{1}{3}}=(\sqrt{8a^3})^{\frac{1}{3}}=\sqrt{2a}.$

7. $(3a\sqrt[3]{3a})^{\frac{1}{4}}=(\sqrt[3]{81a^4})^{\frac{1}{4}}=\sqrt[3]{3a}.$

8. $(\frac{1}{2}\sqrt{2a})^{\frac{1}{4}}=(\sqrt{\frac{1}{2}a})^{\frac{1}{4}}=\sqrt[8]{\frac{1}{2}a}.$

9. $(4c^2\sqrt{2c})^{\frac{1}{5}}=(\sqrt{32c^5})^{\frac{1}{5}}=\sqrt{2c}.$

10. $(4x\sqrt[4]{4x})^{\frac{1}{5}}=(\sqrt[4]{4^5x^5})^{\frac{1}{5}}=\sqrt[4]{4x}.$

11. $\left(\frac{a}{2}\sqrt{\frac{a}{2}}\right)^{\frac{1}{3}}=\left(\sqrt{\frac{a^3}{8}}\right)^{\frac{1}{3}}=\sqrt{\frac{a}{2}}=\frac{1}{2}\sqrt{2a}.$

12. $\left(\frac{a}{3}\sqrt[3]{\frac{a}{3}}\right)^{\frac{1}{2}}=\left(\sqrt[3]{\frac{a^4}{81}}\right)^{\frac{1}{2}}=\sqrt[3]{\frac{a^2}{9}}=\frac{1}{3}\sqrt[3]{3a^2}.$

13. $\left(\frac{a}{4}\sqrt[3]{\frac{a}{4}}\right)^{\frac{1}{4}}=\left(\sqrt[3]{\frac{a^4}{256}}\right)^{\frac{1}{4}}=\sqrt[3]{\frac{a}{4}}=\frac{1}{2}\sqrt[3]{2a}.$

Art. 267. (page 197.)

4. $\dfrac{1}{1+\sqrt{3}}\times\dfrac{\sqrt{3}-1}{\sqrt{3}-1}=\dfrac{\sqrt{3}-1}{2}.$

5. $\dfrac{\sqrt{3}}{3-\sqrt{3}}\times\dfrac{3+\sqrt{3}}{3+\sqrt{3}}=\dfrac{3\sqrt{3}+3}{9-3}=\dfrac{\sqrt{3}+1}{2}.$

6. $\dfrac{a}{\sqrt{x}-\sqrt{y}}\times\dfrac{\sqrt{x}+\sqrt{y}}{\sqrt{x}+\sqrt{y}}=\dfrac{a(\sqrt{x}+\sqrt{y})}{x-y}.$

7. $\dfrac{\sqrt{a}+\sqrt{c}}{\sqrt{a}-\sqrt{c}}\times\dfrac{\sqrt{a}+\sqrt{c}}{\sqrt{a}+\sqrt{c}}=\dfrac{a+2\sqrt{ac}+c}{a-c}$

ADDITION AND SUBTRACTION OF IMAGINARY QUANTITIES.

Art. 270. (page 199.)

2. $\sqrt{-a^2}=a\sqrt{-1}$

$\sqrt{-c^2}=c\sqrt{-1}$

$\text{Sum}=(a+c)\sqrt{-1}$

3. $\sqrt{-4}=2\sqrt{-1}$

$\sqrt{-9}=3\sqrt{-1}$

$\text{Sum}=5\sqrt{-1}$

4. $\sqrt{-8}=2\sqrt{-2}$

$\sqrt{-18}=3\sqrt{-2}$

$\text{Sum}=5\sqrt{-2}$

5. $\sqrt{-16}=4\sqrt{-1}$

$\sqrt{-4}=2\sqrt{-1}$

$\text{Difference}=2\sqrt{-1}$

6. $\sqrt{-8m^2}=2m\sqrt{-2}$

$\sqrt{-2m^2}=m\sqrt{-2}$

$\text{Difference}=m\sqrt{-2}$

MULTIPLICATION OF IMAGINARY QUANTITIES.

2. $\sqrt{-3}=\sqrt{3}\times\sqrt{-1}$

$2\sqrt{-2}=2\sqrt{2}\times\sqrt{-1}$

$\text{Product}=2\sqrt{6}\times-1=-2\sqrt{6}$

3. $a\sqrt{-b^2}=ab\sqrt{-1}$

$2\sqrt{-b^2}=2b\sqrt{-1}$

$\text{Product}=2ab^2\times-1=-2ab^2$

4. $1+\sqrt{-1}$

$1-\sqrt{-1}$

$1+\sqrt{-1}$

$-\sqrt{-1}-(\sqrt{-1})^2$

$1-(-1)=2$

5. $1+\sqrt{-1}$

$1+\sqrt{-1}$

$1+\sqrt{-1}$

$\sqrt{-1}+(\sqrt{-1})^2$

$1+2\sqrt{-1}-1=2\sqrt{-1}$

DIVISION OF IMAGINARY QUANTITIES.

2. $\dfrac{6\sqrt{-3}}{2\sqrt{-4}}=3\sqrt{\tfrac{3}{4}}=\tfrac{3}{2}\sqrt{3}.$

3. $\dfrac{4\sqrt{-a^2}}{a\sqrt{-2}}=\dfrac{4}{a}\sqrt{\dfrac{a^2}{2}}=2\sqrt{2}.$

4. $\dfrac{a\sqrt{-6c}}{\sqrt{-2ac}}=a\sqrt{\dfrac{3}{a}}=\sqrt{3a}.$

5. $\dfrac{2\sqrt{-1}}{1+\sqrt{-1}}=\dfrac{1+2\sqrt{-1}-1}{1+\sqrt{-1}}=1+\sqrt{-1}.$

Art. 273. (page 201.)

6. Given, $\sqrt{(x+5)}=\sqrt{x}+1$. Squaring, $x+5=x+2\sqrt{x}+1$; transposing and reducing, $2\sqrt{x}=4$; dividing and squaring, $x=4$.

7. Given, $3+\sqrt{(2x+4)}=7$. Transposing and reducing, $\sqrt{(2x+4)}=4$; squaring, $2x+4=16$; whence, $x=6$.

8. Given, $8-\sqrt{x}=\sqrt{(x-16)}$. Squaring, $64-16\sqrt{x}+x=x-16$; transposing and reducing, $-16\sqrt{x}=-80$; whence, $x=25$.

9. Given, $\sqrt{(6+\sqrt[3]{3x})}+5=8$. Transposing, $\sqrt{(6+\sqrt[3]{3x})}=3$; squaring, $6+\sqrt[3]{3x}=9$; transposing, $\sqrt[3]{3x}=3$; cubing, $3x=27$; whence, $x=9$.

10. Given, $\sqrt{x}-2=\sqrt{(x-24)}$. Squaring, $x-4\sqrt{x}+4=x-24$; transposing, $-4\sqrt{x}=-28$; whence, $x=49$.

11. Given, $\sqrt{(x+2)}=\dfrac{5}{\sqrt{(x+2)}}$. Clearing of fractions, $x+2=5$; whence, $x=3$.

12. Given, $\sqrt{(x-9)}+\sqrt{(x+11)}=10$. Transposing, $\sqrt{(x-9)}=10-\sqrt{(x+11)}$; squaring, $x-9=100-20\sqrt{(x+11)}+x+11$; transposing and reducing, $20\sqrt{(x+11)}=120$; dividing by 20, $\sqrt{(x+11)}=6$; squaring, $x+11=36$; whence, $x=25$.

13. Given, $\sqrt{(x-a)}=\sqrt{x}-\tfrac{1}{2}\sqrt{a}$. Squaring, $x-a=x-\sqrt{ax}+\tfrac{1}{4}a$; transposing and uniting, $\sqrt{ax}=\dfrac{5a}{4}$; squaring, $ax=\dfrac{25a^2}{16}$; whence, $x=\dfrac{25a}{16}$.

14. Given, $\dfrac{x-2}{\sqrt{x}}=\dfrac{2\sqrt{x}}{3}$. Clearing of fractions, $3x-6=2x$; whence, $x=6$.

15. Given, $\sqrt{(x+4ab)}=2a-\sqrt{x}$. Squaring, $x+4ab=4a^2-4a\sqrt{x}+x$; transposing and reducing, $4a\sqrt{x}=4a^2-4ab$; dividing, $\sqrt{x}=a-b$; whence, $x=(a-b)^2$.

16. Given, $x+\sqrt{(a-x)}=\dfrac{a}{\sqrt{(a-x)}}$. Clearing of fractions, $x\sqrt{(a-x)}+a-x=a$; transposing and reducing, $x\sqrt{(a-x)}=x$; dividing, $\sqrt{(a-x)}=1$; squaring, $a-x=1$; whence, $x=a-1$.

17. Given, $\sqrt{(x-a)}+\sqrt{(x-b)}=\sqrt{(a-b)}$. Squaring, $x-a+2\sqrt{(x^2-ax-bx+ab)}+x-b=a-b$; transposing and reducing, $2\sqrt{(x^2-ax-bx+ab)}=2(a-x)$; dividing by 2 and squaring, $x^2-ax-bx+ab=a^2-2ax+x^2$; transposing and reducing, $ax-bx=a^2-ab$; whence, $x=a$.

18. Given, $\dfrac{x-ax}{\sqrt{x}}=\dfrac{\sqrt{x}}{x}$. Clearing of fractions, $x^2-ax^2=x$; dividing by x, $x-ax=1$; whence, $x=\dfrac{1}{1-a}$.

QUADRATIC EQUATIONS.

Art. 278. (page 204.)

7. Given, $x^2+a^2+b^2=2ab+2x^2$. Transposing, $x^2=a^2-2ab+b^2$; extracting square root, $x=\pm(a-b)$.

8. Given, $4x+8=(x+2)^2$. Expanding, $4x+8=x^2+4x+4$; transposing and reducing, $x^2=4$; extracting square root, $x=\pm 2$.

9. Given, $\dfrac{1}{1-x}+\dfrac{1}{1+x}=3$. Clearing of fractions, $1+x+1-x=3-3x^2$; transposing and uniting, $3x^2=1$; whence, $x^2=\frac{1}{3}$; and $x=\pm\sqrt{\frac{1}{3}}=\pm\frac{1}{3}\sqrt{3}$.

10. Given, $\dfrac{4}{x-3}-\dfrac{4}{x+3}=\frac{1}{3}$. Clearing of fractions, $12x+36-12x+36=x^2-9$; transposing and reducing, $x^2=81$; whence, $x=\pm 9$.

11. Given, $2x+5x^{-1}=3x-11x^{-1}$. Transposing and reducing, $16x^{-1}=x$; multiplying by x, $16=x^2$; whence, $x=\pm 4$.

12. Given, $\dfrac{x+3}{x-3}+\dfrac{x-3}{x+3}=3\frac{1}{3}$. Clearing of fractions, $3x^2+18x+27+3x^2-18x+27=10x^2-90$; transposing and reducing, $4x^2=144$; extracting the square root, $2x=\pm 12$; whence, $x=\pm 6$.

13. Given, $\frac{x}{x+1}+\frac{x}{x+4}=1$. Clearing of fractions, $x^2+4x+x^2+x=x^2+5x+4$; transposing and reducing, $x^2=4$; whence, $x=\pm 2$.

14. Given, $\frac{x}{a}+\frac{a}{x}=\frac{x}{b}+\frac{b}{x}$. Clearing of fractions, $bx^2+a^2b=ax^2+ab^2$; transposing, $(b-a)x^2=(b-a)ab$; whence, $x=\pm\sqrt{ab}$.

15. Given, $(n-x)^3+(n+x)^3=3n^3$. Expanding, $n^3-3n^2x+3nx^2-x^3+n^3+3n^2x+3nx^2+x^3=3n^3$; transposing and uniting, $6nx^2=n^3$; whence, $x^2=\frac{n^2}{6}$; and $x=\pm\frac{n}{6}\sqrt{6}$.

Art. 279. (page 205.)

4. Given, $(x+a)^{\frac{1}{2}}=\frac{a}{(x-a)^{\frac{1}{2}}}$. Squaring, $x+a=\frac{a^2}{x-a}$; clearing of fractions, $x^2-a^2=a^2$; transposing, $x^2=2a^2$; whence, $x=\pm a\sqrt{2}$.

5. Given, $\sqrt{(x+m)}=\sqrt{\{x+\sqrt{(n^2+x^2)}\}}$. Squaring, $x+m=x+\sqrt{(n^2+x^2)}$; transposing, $m=\sqrt{(n^2+x^2)}$; squaring, $m^2=n^2+x^2$; transposing, $x^2=m^2-n^2$; whence, $x=\pm\sqrt{(m^2-n^2)}$.

6. Given, $\sqrt{(x+a)}=\frac{b}{\sqrt{(x-a)}}$. Clearing of fractions, $\sqrt{(x^2-a^2)}=b$; squaring, $x^2-a^2=b^2$; transposing, $x^2=b^2+a^2$; whence, $x=\pm\sqrt{(a^2+b^2)}$.

7. Given, $\sqrt{\{x^2+2ax+\sqrt{(x^2-4)}\}}=a+x$. Squaring, $x^2+2ax+\sqrt{(x^2-4)}=a^2+2ax+x^2$; transposing, $\sqrt{(x^2-4)}=a^2$; squaring, $x^2-4=a^4$; whence, $x=\pm\sqrt{(a^4+4)}$.

8. Given, $\sqrt{\{x^2+\sqrt{(x^4-n^4)}\}}=n$. Squaring, $x^2+\sqrt{(x^4-n^4)}=n^2$; transposing, $\sqrt{(x^4-n^4)}=n^2-x^2$; squaring, $x^4-n^4=n^4-2n^2x^2+x^4$; transposing, $2n^2x^2=2n^4$; whence, $x=\pm n$.

9. Given, $\sqrt{(x+m)}=\sqrt[4]{(x^2+n^2)}$. Raising to fourth power, $x^2+2mx+m^2=x^2+n^2$; transposing, $2mx=n^2-m^2$; whence, $x=\frac{n^2-m^2}{2m}$.

10. Given, $x+\sqrt{(a^2+x^2)}=\frac{2a^2}{\sqrt{(a^2+x^2)}}$. Clearing of fractions, $x\sqrt{(a^2+x^2)}+a^2+x^2=2a^2$; transposing, $x\sqrt{(a^2+x^2)}=a^2-x^2$; squaring, $a^2x^2+x^4=a^4-2a^2x^2+x^4$; transposing, $3a^2x^2=a^4$; whence, $x=\pm\frac{a}{3}\sqrt{3}$.

Art. 280. (page 206.)

3. Let $7\frac{1}{2}+x=$ the larger number; and $7\frac{1}{2}-x=$ the smaller number.
Then, $(7\frac{1}{2}+x)(7\frac{1}{2}-x)=54$; expanding, $56\frac{1}{4}-x^2=54$; transposing, $x^2=2\frac{1}{4}$; whence, $x=\pm 1\frac{1}{2}$; then, $7\frac{1}{2}+x=9$; and $7\frac{1}{2}-x=6$.

4. Let $6+x=$ the larger number; and $6-x=$ the smaller number.
Then, $(6+x)^2+(6-x)^2=74$; expanding, $36+12x+x^2+36-12x+x^2=74$; collecting and transposing, $2x^2=2$; whence, $x^2=1$; and $x=\pm 1$; then, $6+x=7$; and $6-x=5$.

5. Let $x+2\frac{1}{2}=$ the larger number; and $x-2\frac{1}{2}=$ the smaller number.
Then, $(x+2\frac{1}{2})(x-2\frac{1}{2})=84$; expanding, $x^2-6\frac{1}{4}=84$; transposing, $x^2=90\frac{1}{4}$; whence, $x=\pm 9\frac{1}{2}$; then, $x+2\frac{1}{2}=12$; $x-2\frac{1}{2}=7$.

6. Let $12+x=$ the greater part; and $12-x=$ the smaller part.
Then, $(12+x)(12-x)=140$; expanding, $144-x^2=140$; whence, $x=\pm 2$; then, $12+x=14$; and $12-x=10$.

7. Let $x+2=$ the larger number; and $x-2=$ the smaller number.
Then, $(x+2)^2+(x-2)^2=208$; expanding, $x^2+4x+4+x^2-4x+4=208$; transposing, $2x^2=200$; whence, $x=\pm 10$; then, $x+2=12$; and $x-2=8$.

8. Let $x=$ the number. Then, $x^2-\frac{x^2}{4}=432$; whence, $\frac{3x^2}{4}=432$; and $x=\pm 24$.

9. Let $x=$ the number. Then, $\frac{x^2}{2}=\frac{4x^2}{9}+72$; clearing of fractions, $9x^2=8x^2+1296$; transposing, $x^2=1296$; whence, $x=\pm 36$.

10. Let $4x=$ one number; and $5x=$ the other number.
Then, $25x^2-16x^2=81$; whence, $9x^2=81$; and $x=\pm 3$; then, $4x=12$; and $5x=15$.

11. Let $x=$ the greater number; and $\frac{48}{x}=$ the lesser number.
Then, $\frac{x^2}{48}=3$; clearing of fractions, $x^2=144$; whence, $x=\pm 12$; and $\frac{48}{x}=4$.

12. Let $5x=$ the length; and $2x=$ the breadth.
Then, $10x^2=640$ rods; whence, $x=\pm 8$; then, $5x=40$; and $2x=16$.

13. Let x = the number. Then, $170-\frac{x^2}{9}=26$; transposing and changing signs, $\frac{x^2}{9}=144$; extracting the square root, $\frac{x}{3}=\pm 12$; and $x=36$.

14. Let x = the number. Then, $\frac{3}{4}(2x)^2=\frac{5}{4}\left(\frac{4x}{3}\right)^2+28$; expanding, $3x^2=\frac{20x^2}{9}+28$; transposing, $\frac{7x^2}{9}=28$; whence, $x=\pm 6$.

15. Let x = the side of larger; and $\frac{3x}{4}$ = the side of smaller.

Then, $x^3-\frac{27x^3}{64}=999$; collecting, $\frac{37x^3}{64}=999$; dividing by 37, $\frac{x^3}{64}=27$; extracting the cube root, $\frac{x}{4}=3$; whence, $x=12$; and $\frac{3x}{4}=9$.

16. Let $3x$ = the number of yards; and $2x$ = price per yard.

Then, $6x^2=216$; whence, $x=6$; then, $3x=18$; and $2x=12$.

17. Let $5x$ = the length; and $4x$ = the breadth; then, $18x$ = the distance round the field; and $\frac{20x^2}{160}$ = the number of acres in the field.

Then, $\frac{20x^2}{160}\times 5x=90x$; reducing, $\frac{5x^3}{8}=90x$; dividing by $5x$, $\frac{x^2}{8}=18$; whence, $x^2=144$; and $x=\pm 12$; then, $5x=60$; and $4x=48$.

18. Let $\frac{1}{x}$ = part remaining each time. Then, $\frac{81}{x}$ = wine in first remainder; and $\frac{81}{x^2}$ = wine in second remainder.

Hence, $\frac{81}{x^2}=36$; whence, $x=\frac{9}{6}=\frac{3}{2}$; $\frac{1}{x}=\frac{2}{3}$, $\therefore$ $\frac{3}{3}-\frac{2}{3}=\frac{1}{3}$, part drawn each time; $\frac{1}{3}$ of $81=27$, etc.

SECOND SOLUTION. Let x = the number of gallons drawn the first time; then, $81-x$ = the number of gallons of wine left; $\frac{81-x}{81}$ = the quantity of wine in one gallon of mixture; $81-x$ = the number of gallons of mixture left; and $\frac{(81-x)x}{81}$ = the quantity of wine drawn the second time; $\frac{(81-x)^2}{81}$ = the quantity of wine left.

Then, $\frac{(81-x)^2}{81}=36$; extracting the square root, $\frac{81-x}{9}=6$; whence, $x=27$; and $\frac{(81-x)x}{81}=18$.

AFFECTED QUADRATICS.

Art. 282. (page 210.)

22. Given, $x^2-3=\dfrac{x-3}{6}$. Transposing, $x^2-\dfrac{x}{6}=2\frac{1}{2}$; completing the square, $x^2-\dfrac{x}{6}+\frac{1}{144}=\frac{361}{144}$; extracting the square root, $x-\frac{1}{12}=\pm\frac{19}{12}$; whence, $x=1\frac{2}{3}$, or $-1\frac{1}{2}$.

23. Given, $x^2+2px=q$. Completing the square, $x^2+2px+p^2=p^2+q$; extracting the root, $x+p=\pm\sqrt{p^2+q}$; whence, $x=-p\pm\sqrt{(p^2+q)}$.

24. Given, $x^2-ax=b$. Completing the square, $x^2-ax+\dfrac{a^2}{4}=b+\dfrac{a^2}{4}$ $=\dfrac{4b+a^2}{4}$; extracting the root, $x-\dfrac{a}{2}=\pm\frac{1}{2}\sqrt{(4b+a^2)}$; whence, x $=\frac{1}{2}(a\pm\sqrt{4b+a^2})$.

25. Given, $x^2-2nx=m^2-n^2$. Completing the square, $x^2-2nx+n^2$ $=m^2$; extracting the root, $x-n=\pm m$; whence, $x=n\pm m$.

26. Given, $x^2-ax-bx=-ab$. Completing the square, $x^2-(a+b)x$ $+\dfrac{(a+b)^2}{4}=\dfrac{(a+b)^2}{4}-ab=\dfrac{(a-b)^2}{4}$; extracting the root, $x-\dfrac{a+b}{2}$ $=\pm\dfrac{a-b}{2}$; whence, $x=a$, or b.

27. Given, $x+\dfrac{1}{x-3}=5$. Clearing of fractions, $x^2-3x+1=5x-15$; transposing and uniting, $x^2-8x=-16$; completing the square, x^2-8x $+16=0$; extracting the root, $x-4=0$; whence, $x=4$.

28. Given, $x=2+\dfrac{5}{4x}$. Multiplying by x, $x^2=2x+\frac{5}{4}$; transposing, $x^2-2x=\frac{5}{4}$; completing the square, $x^2-2x+1=\frac{9}{4}$; extracting the root, $x-1=\pm\frac{3}{2}$; whence, $x=2\frac{1}{2}$, or $-\frac{1}{2}$.

29. Given, $\dfrac{x-1}{x-3}+2x=12$. Clearing of fractions, $x-1+2x^2-6x$ $=12x-36$; transposing and uniting, $2x^2-17x=-35$; dividing by 2, $x^2-\dfrac{17x}{2}=-\frac{35}{2}$; completing the square, $x^2-\dfrac{17x}{2}+\frac{289}{16}=\frac{9}{16}$; extracting the root, $x-\frac{17}{4}=\frac{3}{4}$; whence, $x=5$, or $3\frac{1}{2}$.

30. Given, $\frac{2}{x+3}+\frac{x+3}{2}=\frac{10}{3}$. Multiplying by $2(x+3)$, $4+x^2+6x+9=\frac{20x}{3}+20$; transposing and uniting, $x^2-\frac{2x}{3}=7$; completing the square, $x^2-\frac{2x}{3}+\frac{1}{9}=\frac{64}{9}$; extracting the root, $x-\frac{1}{3}=\pm\frac{8}{3}$; whence, $x=3$, or $-2\frac{1}{3}$.

31. Given, $\frac{2x}{x+2}+\frac{x+2}{2x}=2$. Clearing of fractions, $4x^2+x^2+4x+4=4x^2+8x$; transposing and uniting, $x^2-4x=-4$; completing the square, $x^2-4x+4=0$; extracting the root, $x-2=0$; whence, $x=2$.

32. Given, $\frac{x}{x+1}+\frac{x+1}{x}=\frac{13}{6}$. Clearing of fractions, $6x^2+6x^2+12x+6=13x^2+13x$; transposing and uniting, $x^2+x=6$; completing the square, $x^2+x+\frac{1}{4}=\frac{25}{4}$; whence, $x=2$, or -3.

33. Given, $\frac{x+2}{x+1}+\frac{x+1}{x+2}=\frac{13}{6}$. Clearing of fractions, $6x^2+24x+24+6x^2+12x+6=13x^2+39x+26$; transposing and uniting, $x^2+3x=4$; completing the square, $x^2+3x+\frac{9}{4}=\frac{25}{4}$; extracting the root, $x+\frac{3}{2}=\pm\frac{5}{2}$; whence, $x=1$, or -4.

34. Given, $\frac{x+1}{x-1}-\frac{x-2}{x+2}=\frac{9}{5}$. Clearing of fractions, $5x^2+15x+10-5x^2+15x-10=9x^2+9x-18$; transposing and uniting, $9x^2-21x=18$; dividing by 9, $x^2-\frac{7x}{3}=2$; completing the square, $x^2-\frac{7x}{3}+\frac{49}{36}=\frac{121}{36}$; extracting the root, $x-\frac{7}{6}=\pm\frac{11}{6}$; whence, $x=3$, or $-\frac{2}{3}$.

35. Given, $x^2+2ax=a^2$. Completing the square, $x^2+2ax+a^2=2a^2$; extracting the root, $x+a=\pm a\sqrt{2}$; whence, $x=a(-1\pm\sqrt{2})$.

36. Given, $3a^2x^{-1}-x=-2a$. Multiplying by x, $3a^2-x^2=-2ax$; transposing, $x^2-2ax=3a^2$; completing the square, $x^2-2ax+a^2=4a^2$; extracting the root, $x-a=\pm 2a$; whence, $x=3a$, or $-a$.

37. Given, $x^2-2ax=b^2-a^2$. Completing the square, $x^2-2ax+a^2=b^2$; extracting the root, $x-a=\pm b$; whence, $x=a+b$, or $a-b$.

38. Given, $x^2-(a-b+c)x=(b-a)c$. Completing the square, $x^2-(a-b+c)x+\frac{(a-b+c)^2}{4}=\frac{(a-b-c)^2}{4}$; extracting the root, $x-\frac{a-b+c}{2}=\pm\frac{a-b-c}{2}$; whence, $x=a-b$, or c.

Art. 285. (page 213.)

11. Given, $2x^2+ax=b$. Multiplying by 8, $16x^2+8ax=8b$; completing the square, $16x^2+8ax+a^2=a^2+8b$; extracting the root, $4x+a=\pm\sqrt{(a^2+8b)}$; whence, $x=\frac{1}{4}\{-a\pm\sqrt{(a^2+8b)}\}$.

12. Given, $\dfrac{x-3a}{b}=\dfrac{9(b-a)}{x}$. Clearing of fractions, $x^2-3ax=9b(b-a)$; multiplying by 4, $4x^2-12ax=36b^2-36ab$; completing the square, $4x^2-12ax+9a^2=36b^2-36ab+9a^2$; extracting the root, $2x-3a=\pm(6b-3a)$; whence, $x=3b$, or $3(a-b)$.

13. Given, $x^2+3x=5$. Multiplying by 4, $4x^2+12x=20$; completing the square, $4x^2+12x+9=29$; extracting the root, $2x+3=\pm\sqrt{29}=\pm 5.3851+$; whence, $x=1.1925+$, or $-4.1925+$.

14. Given, $x^2+2x=5$. Multiplying by 4, $4x^2+8x=20$; completing the square, $4x^2+8x+4=24$; extracting the root, $2x+2=\pm\sqrt{24}=\pm 4.898+$; whence, $x=1.449+$, or $-3.449+$.

15. Given, $x^2-8x=-8$. Multiplying by 4, $4x^2-32x=-32$; completing the square, $4x^2-32x+64=32$; extracting the root, $2x-8=\pm\sqrt{32}=\pm 5.656+$; whence, $x=6.828+$, or $1.172+$.

16. Given, $5x^2-4x=2$. Multiplying by 20, $100x^2-80x=40$; completing the square, $100x^2-80x+16=56$; extracting the root, $10x-4=\pm\sqrt{56}$, or $7.483+$; whence, $x=1.148+$, or $0.348+$.

Art. 289. (page 215.)

6. Given, $x^2+4x^{-2}=5$. Multiplying by x^2, $x^4+4=5x^2$; transposing, $x^4-5x^2=-4$; completing the square, $x^4-5x^2+\frac{25}{4}=\frac{9}{4}$; extracting the square root, $x^2-\frac{5}{2}=\pm\frac{3}{2}$; whence, $x^2=4$, or 1; and $x=\pm 2$, or ± 1.

7. Given, $x+3\sqrt{x}=18$. Completing the square, $x+3\sqrt{x}+\frac{9}{4}=\frac{81}{4}$; extracting the square root, $\sqrt{x}+\frac{3}{2}=\pm\frac{9}{2}$; whence, $\sqrt{x}=3$, or -6; and $x=9$, or 36.

8. Given, $x^{\frac{2}{3}}+4x^{\frac{1}{3}}=5$. Completing the square, $x^{\frac{2}{3}}+4x^{\frac{1}{3}}+4=9$; extracting the square root, $x^{\frac{1}{3}}+2=\pm 3$; hence, $x^{\frac{1}{3}}=1$, or -5; and $x=1$, or -125.

9. Given, $x^{2n}-ax^n=b.$

Completing the square, $x^{2n}-ax^n+\frac{a^2}{4}=b+\frac{a^2}{4};$

Extracting the root, $x^n-\frac{a}{2}=\pm\sqrt{\left(b+\frac{a^2}{4}\right)};$

whence, $x=\left\{\frac{1}{2}a\pm\sqrt{\left(b+\frac{a^2}{4}\right)}\right\}^{\frac{1}{n}}.$

10. Given, $x^{2n}+4x^n=12$. Completing the square, $x^{2n}+4x^n+4=16$; extracting the square root, $x^n+2=\pm4$; whence, $x=\sqrt[n]{2}$, or $\sqrt[n]{-6}$.

11. Given, $x^3+7x^{\frac{3}{2}}=3\frac{3}{4}$. Completing the square, $x^3+7x^{\frac{3}{2}}+\frac{49}{4}=16$; extracting the square root, $x^{\frac{3}{2}}+\frac{7}{2}=\pm4$; whence, $x^{\frac{3}{2}}=\frac{1}{2}$, or $-7\frac{1}{2}$; then, $x^3=\frac{1}{4}$, or $56\frac{1}{4}$; and $x=\frac{1}{2}\sqrt[3]{2}$, or $\frac{1}{2}\sqrt[3]{450}$.

12. Given, $\sqrt[3]{x}+2\sqrt[3]{x^2}=\frac{3}{8}$. Transposing and dividing by 2, $x^{\frac{2}{3}}+\frac{1}{2}x^{\frac{1}{3}}=\frac{3}{16}$; completing the square, $x^{\frac{2}{3}}+\frac{1}{2}x^{\frac{1}{3}}+\frac{1}{16}=\frac{1}{4}$; extracting the square root, $x^{\frac{1}{3}}+\frac{1}{4}=\pm\frac{1}{2}$; whence, $x^{\frac{1}{3}}=\frac{1}{4}$, or $-\frac{3}{4}$; and $x=\frac{1}{64}$, or $-\frac{27}{64}$.

13. Given, $x^n-ax^{\frac{n}{2}}=b.$

Completing the square, $x^n-ax^{\frac{n}{2}}+\frac{a^2}{4}=b+\frac{a^2}{4};$

Extracting the square root, $x^{\frac{n}{2}}-\frac{a}{2}=\pm\sqrt{\left(b+\frac{a^2}{4}\right)};$

whence, $x^{\frac{n}{2}}=\frac{a}{2}\pm\sqrt{\left(b+\frac{a^2}{4}\right)};$

$$x=\left\{\frac{1}{2}a\pm\sqrt{\left(b+\frac{a^2}{4}\right)}\right\}^{\frac{2}{n}}.$$

14. Given, $\frac{\sqrt{4x}+2}{4+\sqrt{x}}=\frac{4+\sqrt{x}}{\sqrt{x}}$. Clearing of fractions, $2x+2\sqrt{x}=16+8\sqrt{x}+x$; transposing and uniting, $x-6\sqrt{x}=16$; completing the square, $x-6\sqrt{x}+9=25$; extracting the square root, $\sqrt{x}-3=\pm5$; whence, $\sqrt{x}=8$, or -2; and $x=64$, or 4.

Art. 291. (page 216.)

6. Given; $\sqrt{(5+x)}+\sqrt[4]{(5+x)}=6$. Completing the square, $\sqrt{(5+x)}+\sqrt[4]{(5+x)}+\frac{1}{4}=\frac{25}{4}$; extracting the square root, $\sqrt[4]{(5+x)}+\frac{1}{2}=\pm\frac{5}{2}$; whence, $\sqrt[4]{(5+x)}=2$, or -3; raising to the fourth power, $5+x=16$, or 81; whence, $x=11$, or 76.

7. Given, $\left(\frac{4}{x}+x\right)^2+6\left(\frac{4}{x}+x\right)=40$. Completing the square, $\left(\frac{4}{x}+x\right)^2+6\left(\frac{4}{x}+x\right)+9=49$; extracting the square root, $\frac{4}{x}+x+3=\pm 7$; whence, $\frac{4}{x}+x=4$, or -10; clearing of fractions and using first value, $4+x^2=4x$; transposing, $x^2-4x=-4$; completing the square, $x^2-4x+4=0$; extracting the square root, $x=2$; clearing of fractions and using second value, $4+x^2=-10x$; transposing, $x^2+10x=-4$; completing the square, $x^2+10x+25=21$; extracting the square root, $x+5=\pm\sqrt{21}$; whence, $x=-5\pm\sqrt{21}$.

8. Given, $x-\sqrt{(x+5)}=1$. Adding 5, $x+5-\sqrt{(x+5)}=6$; completing the square, $x+5-\sqrt{(x+5)}+\frac{1}{4}=\frac{25}{4}$; extracting the square root, $\sqrt{(x+5)}-\frac{1}{2}=\pm\frac{5}{2}$; whence, $\sqrt{(x+5)}=3$, or -2; then, $x+5=9$, or 4; and $x=4$, or -1.

9. Given, $(x-4)^2-6\sqrt{(x-4)}=\frac{16}{x-4}$. Clearing of fractions, $(x-4)^3-6(x-4)^{\frac{3}{2}}=16$; completing the square, $(x-4)^3-6(x-4)^{\frac{3}{2}}+9=25$; extracting the square root, $(x-4)^{\frac{3}{2}}+3=\pm 5$; whence, $(x-4)^{\frac{3}{2}}=2$, or -8; then, $x-4=2^{\frac{2}{3}}$, or 4; $x=4+\sqrt[3]{4}$, or 8.

10. Given, $(x^2+2x-3)^2+7(x^2+2x-3)=60$. Completing the square, $(x^2+2x-3)^2+7(x^2+2x-3)+\frac{49}{4}=\frac{289}{4}$; extracting the square root, $x^2+2x-3+\frac{7}{2}=\pm\frac{17}{2}$; whence, $x^2+2x=8$, or -9; completing the square with first value, $x^2+2x+1=9$; extracting the square root, $x+1=\pm 3$; whence, $x=2$, or -4; completing the square with second value, $x^2+2x+1=-8$; extracting the square root, $x+1=\pm 2\sqrt{-2}$; whence, $x=-1\pm 2\sqrt{-2}$.

11. Given, $(x^2-9)^2-11x^2+40=21$. Adding 59 and factoring, $(x^2-9)^2-11(x^2-9)=80$; completing the square, $(x^2-9)^2-11(x^2-9)+\frac{121}{4}=\frac{441}{4}$; extracting the square root, $x^2-9-\frac{11}{2}=\pm\frac{21}{2}$; whence, $x^2=25$, or 4; and $x=\pm 5$, or ± 2.

12. Given, $(x^2-4x+5)^2+4x^2-16x=-8$. Adding 20 and factoring, $(x^2-4x+5)^2+4(x^2-4x+5)=12$; completing the square, $(x^2-4x+5)^2+4(x^2-4x+5)+4=16$; extracting the square root, $x^2-4x+5+2=\pm 4$; whence, $x^2-4x=-3$, or -11; completing the square, $x^2-4x+4=1$, or -7; extracting the square root, $x-2=\pm 1$, or $\pm\sqrt{-7}$; whence, $x=3$, or 1, or $2\pm\sqrt{-7}$.

PROBLEMS PRODUCING AFFECTED QUADRATICS.

4. Let x = first one's share; and $50-x$ = second one's share.

Then, $x(50-x)=600$; expanding and changing signs, $x^2-50x=-600$; completing the square, $x^2-50x+625=25$; extracting the root, $x-25=\pm 5$; whence, $x=30$, or 20; and $50-x=20$, or 30.

5. Let x = length; $\dfrac{1008}{x}$ = breadth.

Then, $2x+\dfrac{2016}{x}=128$; clearing of fractions, $2x^2+2016=128x$; transposing and dividing by 2, $x^2-64x=-1008$; completing the square, $x^2-64x+1024=16$; extracting the root, $x-32=\pm 4$; whence, $x=36$, or 28; and $\dfrac{1008}{x}=28$, or 36.

Although both these answers are positive, yet only the first fulfills the condition of the question, since the length must be the longest side. To make the second results true, we must take x equal to the breadth.

6. Let x = the number in file; and $x+60$ = the number in rank.

Then, $x^2+60x=1600$; completing the square, $x^2+60x+900=2500$; extracting the square root, $x+30=\pm 50$; whence, $x=20$, or -80; and $x+60=80$, or -20.

If the example read, each *file* exceeds each *rank* by 60, the second results will be arithmetically true.

7. Let x = number of Bibles; $\dfrac{5000}{x}$ = price of one Bible.

Then, $550x-5000=\dfrac{5000}{x}$; clearing of fractions, $550x^2-5000x=5000$; dividing by 550, $x^2-\frac{100}{11}x=\frac{100}{11}$; completing the square, $x^2-\frac{100}{11}x+\frac{2500}{121}=\frac{3600}{121}$; extracting the square root, $x-\frac{50}{11}=\pm\frac{60}{11}$; whence, $x=10$, or $-\frac{10}{11}$.

8. Let x = the length; and $24-x$ = the breadth.

Then, $(24-x)x=35(x-24+x)$; expanding, $24x-x^2=70x-840$; transposing and uniting, $x^2+46x=840$; completing the square, $x^2+46x+529=1369$; extracting the root, $x+23=\pm 37$; whence, $x=14$, or -60; and $24-x=10$, or 84.

9. Let x = A's number; and $10-x$ = B's number; then, $\frac{12}{x}$ = price of one of A's; and $\frac{12}{10-x}$ = price of one of B's.

Then, $\frac{12}{x}-\frac{12}{10-x}=1$; clearing of fractions, $120-12x-12x=10x-x^2$; transposing and uniting, $x^2-34x=-120$; completing the square, $x^2-34x+289=169$; extracting the square root, $x-17=\pm 13$; whence, $x=30$, or 4; and $10-x=-20$, or 6.

The first pair of answers will be arithmetically correct if the question reads thus: A bought some oranges and B sold some, when they had 10 more than at first; A paid 12 cents and B received 12 cents, and A's oranges cost one cent more apiece than B's; how many oranges did A buy and B sell?

10. Let x = the number of sheep; then, $\frac{180}{x}$ = original price per head; and $\frac{180}{x-2}$ = price per head by second condition.

Then, $\frac{180}{x-2}-\frac{180}{x}=1$; clearing of fractions, $180x-180x+360=x^2-2x$; transposing and uniting, $x^2-2x=360$; completing the square, $x^2-2x+1=361$; extracting the square root, $x-1=\pm 19$; whence, $x=20$, or -18.

Had he bought two more for the same money, they would have cost $1 per head less. These conditions make the second answer arithmetically true.

11. Let x = the number of miles he traveled per hour; then, $\frac{48}{x}$ = the time required; and $\frac{48}{x+4}$ = the time required by second condition.

Then, $\frac{48}{x}-\frac{48}{x+4}=6$; clearing of fractions, $48x+192-48x=6x^2+24x$; reducing and dividing by 6, $x^2+4x=32$; completing the square, $x^2+4x+4=36$; extracting the square root, $x+2=\pm 6$; whence, $x=4$, or -8.

12. Let x = the number of rows; and $x+3$ = the number of trees.

Then, $x(x+3)=180$; expanding, $x^2+3x=180$; completing the square, $x^2+3x+\frac{9}{4}=\frac{729}{4}$; extracting the square root, $x+\frac{3}{2}=\pm\frac{27}{2}$; whence, $x=12$, or -15; and $x+3=15$, or -12.

13. Let x = number coins of silver; and $52-x$ = number coins of copper; then, $x(52-x)$ = value of the silver; and $x(52-x)$ = value of the copper.

Then, $2x(52-x)=200$; expanding and dividing by -2, $x^2-52x=-100$; completing the square, $x^2-52x+676=576$; extracting the square root, $x-26=\pm 24$; whence, $x=2$, or 50; and $52-x=50$, or 2.

Although both values of x are arithmetically true, the conditions of the question require the first to be used, since we do not have silver 2-cent pieces and copper 50-cent pieces.

14. Let x = the number of paupers; then, $\dfrac{600}{x}$ = what each received; and $\dfrac{600}{x+5}$ = what he expected them to receive.

Then, $\dfrac{600}{x}-\dfrac{600}{x+5}=10$; clearing of fractions, $600x+3000-600x=10x^2+50x$; transposing and dividing by 10, $x^2+5x=300$; completing the square, $x^2+5x+\frac{25}{4}=\frac{1225}{4}$; extracting the square root, $x+\frac{5}{2}=\pm\frac{35}{2}$; whence, $x=15$, or -20.

15. Let x = the number of persons; and $x+30$ = what each pays.
Then, $x(x+30)=1000$; whence, $x=20$, or -50.

16. Let x = the first digit; and $10-x$ = the second digit.
Then, $x^2+100-20x+x^2=52$; whence, $x=4$, or 6; $10-x=6$, or 4.

17. Let x = cost of the horse; then, $\dfrac{x}{100}$ = gain per cent.; and $x\left(\dfrac{x}{100}\right)$ = gain.

Then, $x+\dfrac{x^2}{100}=171$; whence, $x=90$, or -190.

18. Let x = the sum laid out; then, $\dfrac{x}{100}$ = the loss per cent.; and $\dfrac{x^2}{100}$ = the loss.

Then, $x-\dfrac{x^2}{100}=24$; whence, $x=40$, or 60.

19. Let x = the rate of sailing; then, $\dfrac{90}{x+3}$ = the number of hours required to go down the river; and $\dfrac{90}{x-3}$ = the number of hours required to return.

Then, $\dfrac{90}{x+3}+\dfrac{90}{x-3}=16$; clearing of fractions, $90x-270+90x+270=16x^2-144$; whence, $x=12$, or $-\frac{3}{4}$.

20. Let x = the number of sheep; $\frac{80}{x}$ = the price per head; $\frac{80}{x+4}$ = the price per head by second condition.

Then, $\frac{80}{x} - \frac{80}{x+4} = 1$; clearing of fractions, $80x + 320 - 80x = x^2 + 4x$; whence, $x = 16$, or -20.

21. Let x = the price per pound; $\frac{216}{x}$ = the number of pounds.

Then, $\frac{216}{x} - \frac{216}{x+1} = 3$; clearing of fractions, $216x + 216 - 216x = 3x^2 + 3x$; whence, $x = 8$, or -9; and $\frac{216}{x} = 27$, or -24.

22. Let x = the width of frame; then, $(18+12)2 + 4x = 60 + 4x$ = the length of frame; and $18 \times 12 = 216$ = the surface of frame.

Then, $(60 + 4x)x = 216$; expanding, $4x^2 + 60x = 216$; whence, $x = 3$, or -18.

23. Let x = the number to be bought for twelve cents; then, $\frac{12}{x}$ = the price of one egg; $\frac{12}{x-2}$ = price of one egg by second condition.

Then, $\frac{144}{x-2} - \frac{144}{x} = 1$; clearing of fractions, $144x - 144x + 288 = x^2 - 2x$; whence, $x = 18$, or -16; and $\frac{144}{x} = 8$.

24. Let x = B's rate of traveling; and $x+1$ = A's rate of traveling; then, $\frac{90}{x}$ = the number of hours B takes; and $\frac{90}{x+1}$ = the number of hours A takes.

Then, $\frac{90}{x} - \frac{90}{x+1} = 1$; clearing of fractions, $90x + 90 - 90x = x^2 + x$; whence, $x = 9$, or -10; and $x + 1 = 10$, or -9.

25. Let x = the number of yards; and $\frac{72}{x}$ = price of one yard.

Then, $6\frac{1}{2}x - 72 = \frac{72}{x}$; clearing of fractions, $13x^2 - 144x = 144$; whence, $x = 12$, or $-\frac{12}{13}$.

Art. 297. (page 221.)

2.

$$xy=15; \quad (1)$$
$$x+y=8. \quad (2)$$

From (1), $y=\dfrac{15}{x}$; (3)

Substituting in (2), $x+\dfrac{15}{x}=8$; (4)

Clearing of fractions, $x^2+15=8x$; (5)

whence, $x=5$, or 3;

and $y=3$, or 5.

3.

$$x-y=3; \quad (1)$$
$$x^2-y^2=21. \quad (2)$$

Dividing (2) by (1), $x+y=7$; (3)

Adding (1) and (3), $x=5$;

whence, $y=2$.

4.

$$x+y=6; \quad (1)$$
$$x^2+y^2=20. \quad (2)$$

From (1), $x=6-y$; (3)

Squaring (3), $x^2=36-12y+y^2$; (4)

Substituting in (2), $36-12y+y^2+y^2=20$;

Reducing, $y^2-6y=-8$;

Completing the square, $y^2-6y+9=1$;

whence, $y=4$, or 2;

and $x=2$, or 4.

5.

$$xy=8; \quad (1)$$
$$4x-3y=10. \quad (2)$$

From (1), $x=\dfrac{8}{y}$; (3)

Substituting in (2), $\dfrac{32}{y}-3y=10$; (4)

Clearing of fractions, $32-3y^2=10y$; (5)

whence, $y=2$, or $-5\frac{1}{3}$;

and $x=4$, or $-1\frac{1}{2}$.

6.

$$2x+y=11; \quad (1)$$
$$3x^2-y^2=2. \quad (2)$$

From (1), $y=11-2x$; (3)

Substituting in (2), $3x^2-121+44x-4x^2=2$; (4)

Reducing, $x^2-44x=-123$;

whence, $x=3$, or 41;

and $y=5$, or -71.

7.
$$xy = 18; \qquad (1)$$
$$3y - 2x = 12. \qquad (2)$$

From 1, $y = \dfrac{18}{x}$; (3)

Substituting in (2), $\dfrac{54}{x} - 2x = 12$;

Clearing of fractions, $54 - 2x^2 = 12x$;

whence, $x = 3$, or -9;

and $y = 6$, or -2.

8.
$$\frac{1}{x} + \frac{1}{y} = 5; \qquad (1)$$
$$\frac{1}{x^2} + \frac{1}{y^2} = 13. \qquad (2)$$

From (1), $\dfrac{1}{y} = 5 - \dfrac{1}{x}$; (3)

Substituting in (2), $\dfrac{1}{x^2} + 25 - \dfrac{10}{x} + \dfrac{1}{x^2} = 13$;

Reducing, $\dfrac{1}{x^2} - \dfrac{5}{x} = -6$;

Completing the square, $\dfrac{1}{x^2} - \dfrac{5}{x} + \frac{25}{4} = \frac{1}{4}$;

whence, $\dfrac{1}{x} = 2$, or 3;

and $x = \frac{1}{2}$, or $\frac{1}{3}$;

then, $y = \frac{1}{3}$, or $\frac{1}{2}$.

9.
$$xy = 35; \qquad (1)$$
$$x^2 - y^2 = 24. \qquad (2)$$

From (1), $x = \dfrac{35}{y}$;

Substituting in (2), $\dfrac{1225}{y^2} - y^2 = 24$;

Clearing of fractions, $1225 - y^4 = 24y^2$;

Completing the square, $y^4 + 24y^2 + 144 = 1369$;

Extracting the square root, $y^2 + 12 = \pm 37$;

whence, $y^2 = 25$, or -49;

and $y = \pm 5$, or $\pm 7\sqrt{-1}$;

also, $x = \pm 7$, or $\mp 5\sqrt{-1}$.

10. $x^3-y^3=28(x-y)$; (1)

$x+y=6.$ (2)

Dividing (1) by $(x-y)$, $x^2+xy+y^2=28$; (3)

From (2), $x=6-y$; (4)

Substituting in (3), $36-12y+y^2+6y-y^2+y^2=28$; (5)

Uniting, $y^2-6y=-8$;

whence, $y=2$, or 4;

and $x=4$, or 2.

Most of these problems may also be solved as shown in Example 1, Case III.

Art. 299. (page 223.)

These problems may all be solved by the method given in the Algebra, but some are capable of shorter solutions, which will be given below.

$x^2-2xy=5$; (1)

$x^2-y^2=21.$ (2)

Let $y=vx$;

Substituting in (1 $x^2-2vx^2=5$; (3)

Substituting in (2), $x^2-v^2x^2=21$; (4)

From (3), $x^2=\dfrac{5}{1-2v}$; (5)

From (4), $x^2=\dfrac{21}{1-v^2}$; (6)

Equating (5) and (6), $\dfrac{5}{1-2v}=\dfrac{21}{1-v^2}$; (7)

Clearing of fractions, $5-5v^2=21-42v$; (8)

Reducing, $v^2-\frac{42}{5}v=-\frac{16}{5}$;

whence, $v=\frac{2}{5}$, or 8;

Substituting in (5), $x^2=\dfrac{5}{1-\frac{4}{5}}$, or $\dfrac{5}{1-16}$;

and $x=\pm 5$, or $\pm\frac{1}{3}\sqrt{-3}$;

whence, $y=\pm 2$, or $\pm\frac{8}{3}\sqrt{-3}$.

3. $x^2-y^2=12$; (1)

$x^2-xy+y^2=12.$ (2)

Equating (1) and (2), $x^2-y^2=x^2-xy+y^2$; (3)

Uniting, $xy=2y^2$;

whence, $x=2y$; (4)

Substituting in (1), $4y^2-y^2=12$;

whence, $y^2=4$;

and $y=\pm 2$;

also $x=\pm 4$.

4.

$$x^2y(x+y)=20; \qquad (1)$$
$$x^2y(2x-3y)=20. \qquad (2)$$

Equating (1) and (2), $x^2y(x+y)=x^2y(2x-3y)$;

whence, $x=4y$; (3)

Substituting in (1), $80y^4=20$;

whence, $y^4=\frac{1}{4}$;

and $y^2=\pm\frac{1}{2}$;

then, $y=\pm\frac{1}{2}\sqrt{2}$, or $\pm\frac{1}{2}\sqrt{-2}$;

and $x=\pm 2\sqrt{2}$, or $\pm 2\sqrt{-2}$.

5.

$$x^2-xy=8; \qquad (1)$$
$$x^2-y^2=12. \qquad (2)$$

Dividing (1) by (2), $\frac{x}{x+y}=\frac{2}{3}$; (3)

Clearing of fractions, $x=2y$; (4)

Substituting in (2), $4y^2-y^2=12$;

whence, $y^2=4$;

and $y=\pm 2$;

also, $x=\pm 4$.

Solving this problem by substituting vx for y, we have

$$x^2-vx^2=8; \qquad (1)$$
$$x^2-v^2x^2=12. \qquad (2)$$

Equating values of x^2, $\frac{8}{1-v}=\frac{12}{1-v^2}$; (3)

Clearing of fractions, $8-8v^2=12-12v$;

whence, $v=1$, or $\frac{1}{2}$;

Substituting the first value, $x^2=\frac{8}{1-1}=\frac{8}{0}=\infty$;

and $x=\pm\sqrt{\infty}$;

Substituting the second value, $x^2=\frac{8}{\frac{1}{2}}=16$;

and $x=\pm 4$;

whence, $y=\pm\sqrt{\infty}$, or ± 2.

6.
$$x^2+xy=10; \qquad (1)$$
$$x^2+y^2=13. \qquad (2)$$

Let $y=vx$;

Substituting in (1), $x^2+vx^2=10$; (3)

Substituting in (2), $x^2+v^2x^2=13$; (4)

Equating values of x^2, $\dfrac{10}{1+v}=\dfrac{13}{1+v^2}$; (5)

Clearing of fractions, $10+10v^2=13+13v$; (6)

Reducing, $v^2-\frac{13}{10}v=\frac{3}{10}$;

whence, $v=\frac{3}{2}$, or $-\frac{1}{5}$;

Substituting first value, $x^2=\dfrac{10}{1+\frac{3}{2}}=4$;

whence, $x=\pm 2$;

Substituting second value, $x^2=\dfrac{10}{1-\frac{1}{5}}=\frac{25}{2}$;

whence, $x=\pm\frac{5}{2}\sqrt{2}$;

and $y=\pm 3$, or $\mp\frac{1}{2}\sqrt{2}$.

7.
$$x^2-y^2=3. \qquad (1)$$
$$x^2-2xy+2y^2=2. \qquad (2)$$

Let $y=vx$;

Substituting in (1), $x^2-v^2x^2=3$; (3)

and in (2), $x^2-2vx^2+2v^2x^2=2$; (4)

Equating values of x^2, $\dfrac{3}{1-v^2}=\dfrac{2}{1-2v+2v^2}$; (5)

Clearing of fractions, $3-6v+6v^2=2-2v^2$;

whence, $v=\frac{1}{2}$, or $\frac{1}{4}$;

Substituting first value, $x^2=\dfrac{3}{1-\frac{1}{4}}=4$;

whence, $x=\pm 2$;

Substituting second value, $x^2=\dfrac{3}{1-\frac{1}{16}}=\frac{16}{5}$;

whence, $x=\pm\frac{4}{5}\sqrt{5}$;

and $y=\pm 1$, or $\pm\frac{1}{5}\sqrt{5}$.

Art. 301. (page 224.)

4. $xy = 20;$ (1)
$x - y = 1.$ (2)

Squaring (2), $x^2 - 2xy + y^2 = 1;$ (3)
Multiplying (1) by (4), $4xy = 80.$ (4)

Adding (3) and (4), $x^2 + 2xy + y^2 = 81;$ (5)
Evolving, $x + y = \pm 9;$ (6)
Adding (2) and (6), $x = 5$, or $-4;$
Subtracting (2) from (6), $y = 4$, or $-5.$

5. $\sqrt{xy} = 2;$ (1)
$\sqrt{x} + \sqrt{y} = 3.$ (2)

Squaring (2), $x + 2\sqrt{xy} + y = 9;$ (3)
Multiplying (1) by (4), $4\sqrt{xy} = 8.$ (4)

Subtracting (4) from (3), $x - 2\sqrt{xy} + y = 1;$ (5)
Evolving, $\sqrt{x} - \sqrt{y} = \pm 1;$ (6)
Adding (2) and (6), $\sqrt{x} = 2$, or $1;$
whence, $x = 4$, or $1;$
Subtracting (6) from (2), $\sqrt{y} = 1$, or $2;$
whence, $y = 1$, or $4.$

6. $x - y = 3;$ (1)
$\frac{x^2}{y^2} + \frac{4x}{y} = 32.$ (2)

Completing the square in (2), $\frac{x^2}{y^2} + \frac{4x}{y} + 4 = 36;$ (3)

Evolving, $\frac{x}{y} + 2 = \pm 6;$ (4)

whence, $\frac{x}{y} = 4$, or $-8;$

and $x = 4y$, or $-8y;$
Substituting first value in (1), $4y - y = 3;$
whence, $y = 1;$
and $x = 4;$
Substituting second value in (1), $-8y - y = 3;$
whence, $y = -\frac{1}{3};$
and $x = 2\frac{2}{3}.$

7.

	$xy=15;$	(1)
	$(x+y)^2-6(x+y)=16.$	(2)
Completing the sq. in (2),	$(x+y)^2-6(x+y)+9=25;$	(3)
Evolving,	$x+y-3=\pm 5;$	
whence,	$x+y=8, \text{ or } -2;$	(4)
Squaring (4),	$x^2+2xy+y^2=64, \text{ or } 4;$	(5)
Multiplying (1) by 4,	$4xy=60.$	(6)
Subtracting (6) from (5),	$x^2-2xy+y^2=4, \text{ or } -56;$	(7)
Evolving,	$x-y=\pm 2, \text{ or } \pm 2\sqrt{-14};$	(8)
Adding (4) and (8),	$x=5, 3, \text{ or } -1\pm\sqrt{-14};$	
Subtracting (8) from (4),	$y=3, 5, \text{ or } -1\mp\sqrt{-14};$	

8.

	$x+\sqrt{xy}+y=9;$	(1)
	$x^2+xy+y^2=27.$	(2)
Dividing (2) by (1),	$x-\sqrt{xy}+y=3;$	(3)
Subtracting (3) from (1),	$\sqrt{xy}=3;$	(4)
whence,	$xy=9;$	(5)
Substituting (4) in (1),	$x+y=6;$	(6)
Multiplying (5) by 3,	$3xy=27;$	(7)
Subtracting (7) from (2),	$x^2-2xy+y^2=0;$	
whence,	$x-y=0;$	
and	$x=y;$	
Substituting in (6),	$x=3;$	
and	$y=3.$	

9.

	$x-y=2;$	(1)
	$x^3-y^3=152;$	(2)
Dividing (2) by (1),	$x^2+xy+y^2=76;$	(3)
Squaring (1),	$x^2-2xy+y^2=4.$	(4)
Subtracting (4) from (3),	$3xy=72;$	
whence,	$xy=24;$	(5)
Adding (5) to (3),	$x^2+2xy+y^2=100;$	(6)
Evolving,	$x+y=\pm 10;$	(7)
Adding (1) and (7),	$x=6, \text{ or } -4;$	
Subtracting (1) from (7),	$y=4, \text{ or } -6.$	

10. $$x^3 - y^3 = 19; \quad (1)$$
$$x^2y - xy^2 = 6. \quad (2)$$

Multiplying (2) by 3, $3x^2y - 3xy^2 = 18$; (3)
Subtracting (3) from (1), $x^3 - 3x^2y + 3xy^2 - y^3 = 1$; (4)
Extracting the cube root, $x - y = 1$; (5)
Dividing (2) by (5), $xy = 6$; (6)
From (5), $x = 1 + y$;
Substituting in (6), $y + y^2 = 6$;
whence, $y = 2$, or -3;
and $x = 3$, or -2.

11. $$\frac{x^2}{y} + \frac{y^2}{x} = 9; \quad (1)$$
$$x + y = 6. \quad (2)$$

Clearing (1) of fractions, $x^3 + y^3 = 9xy$; (3)
Dividing (3) by (2), $x^2 - xy + y^2 = \frac{3}{2}xy$; (4)
Squaring (2), $x^2 + 2xy + y^2 = 36$. (5)
Subtracting (4) from (5), $3xy = 36 - \frac{3}{2}xy$;
Transposing, $4\frac{1}{2}xy = 36$;
whence, $xy = 8$; (6)
Subtracting (6) from (4), $x^2 - 2xy + y^2 = \frac{3}{2}xy - 8 = 4$;
Evolving, $x - y = \pm 2$; (7)
Adding (7) and (2), $x = 4$, or 2;
whence, $y = 2$, or 4.

Art. 303. (page 226.)

3. $$x^2 + y^2 + 2x = 19; \quad (1)$$
$$xy + y = 8. \quad (2)$$

Multiplying (2) by 2, $2xy + 2y = 16$ (3)
Adding (1) and (3), $x^2 + 2xy + y^2 + 2(x + y) = 35$; (4)
Completing the square, $(x + y)^2 + 2(x + y) + 1 = 36$; (5)
Evolving, $x + y + 1 = \pm 6$;
and $x + y = 5$, or -7; (6)
From (6), $x = 5 - y$, or $-7 - y$;
Substituting in (2), $5y - y^2 + y = 8$;
whence, $y = 4$, or 2;
and $x = 1$, or 3;
Substituting 2d value in (2), $-7y - y^2 + y = 8$.
whence, $y = -2$, or -4;
and $x = -5$, or -3.

4.
$$x-3y=3; \quad (1)$$
$$x^2+9y^2=45. \quad (2)$$

Squaring (1), $x^2-6xy+9y^2=9$; (3)

Subtracting (3) from (2), $6xy=36$; (4)

Adding (2) and (4), $x^2+6xy+9y^2=81$; (5)

Extracting square root, $x+3y=\pm 9$; (6)

Adding (1) and (6), $x=6$, or -3;

Subtracting (1) from (6), $y=1$, or -2.

5.
$$xy=ab; \quad (1)$$
$$\frac{x}{a}+\frac{y}{b}=2. \quad (2)$$

From (1), $x=\dfrac{ab}{y}$; (3)

Substituting in (2), $\dfrac{b}{y}+\dfrac{y}{b}=2$; (4)

Clearing of fractions, $b^2+y^2=2by$; (5)

Completing the square, $y^2-2by+b^2=0$; (6)

Extracting the square root, $y=b$;

and $x=a$.

6.
$$x^2+y^2=20; \quad (1)$$
$$xy-x-y=2. \quad (2)$$

Multiplying (2) by 2, $2xy-2x-2y=4$; (3)

Adding (1) and (3), $x^2+2xy+y^2-2(x+y)=24$; (4)

Completing the square, $(x+y)^2-2(x+y)+1=25$; (5)

Extracting the square root, $x+y-1=\pm 5$; (6)

whence, $x+y=6$, or -4; (7)

Substituting first value in (2), $xy-6=2$;

whence, $xy=8$;

and $2xy=16$; (8)

Subtracting (8) from (1), $x^2-2xy+y^2=4$;

Extracting the square root, $x-y=\pm 2$; (9)

Adding (7) and (9), $x=4$, or 2;

Subtracting (9) from (7), $y=2$, or 4.

Other values may be found by using the second value of $x+y$. This will be found to be the case in other examples where the evolution is performed twice.

7.
$$\frac{x}{a}+\frac{y}{b}=1; \qquad (1)$$
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1. \qquad (2)$$

Squaring (1), $\frac{x^2}{a^2}+\frac{2xy}{ab}+\frac{y^2}{b^2}=1;$ (3)

Subtracting (2) from (3), $\frac{2xy}{ab}=0;$ (4)

Subtracting (4) from (2), $\frac{x^2}{a^2}-\frac{2xy}{ab}+\frac{y^2}{b^2}=1;$ (5)

Extracting the square root, $\frac{x}{a}-\frac{y}{b}=\pm 1;$ (6)

Adding (1) and (6), $\frac{x}{a}=1$, or 0;

whence, $x=a$, or 0;

and $y=0$, or b.

8.
$$x^2+y^2=106; \qquad (1)$$
$$x-y+\sqrt{(x-y)}=6. \qquad (2)$$

Completing the sq. of (2), $x-y+\sqrt{(x-y)}+\frac{1}{4}=\frac{25}{4};$

Extracting the square root, $\sqrt{(x-y)}+\frac{1}{2}=\pm\frac{5}{2};$

whence, $\sqrt{(x-y)}=2$, or -3;

and $x-y=4$, or 9; (3)

Squaring (3), $x^2-2xy+y^2=16$, or 81; (4)

Subtracting (3) from (1), $2xy=90$, or 25; (5)

Adding (5) to (1), $x^2+2xy+y^2=196$, or 131;

Extracting the square root, $x+y=\pm 14$, or $\pm\sqrt{131}$; (6)

Adding (3) and (6), $x=9$, or -5, or $4\frac{1}{2}\pm\frac{1}{2}\sqrt{131}$;

and $y=5$, or -9, or $-4\frac{1}{2}\pm\frac{1}{2}\sqrt{131}$.

9.
$$x+y=a^3-b^3; \qquad (1)$$
$$x^{\frac{1}{3}}+y^{\frac{1}{3}}=a-b. \qquad (2)$$

Dividing (1) by (2), $x^{\frac{2}{3}}-x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}}=a^2+ab+b^2;$ (3)

Squaring (2), $x^{\frac{2}{3}}+2x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}}=a^2-2ab+b^2.$ (4)

Subtracting (3) from (4), $3x^{\frac{1}{3}}y^{\frac{1}{3}}=-3ab;$ (5)

Dividing (5) by 3, $x^{\frac{1}{3}}y^{\frac{1}{3}}=-ab;$ (6)

Subtracting (6) from (3), $x^{\frac{2}{3}}-2x^{\frac{1}{3}}y^{\frac{1}{3}}+y^{\frac{2}{3}}=a^2+2ab+b^2;$

Extracting square root, $x^{\frac{1}{3}}-y^{\frac{1}{3}}=\pm(a+b);$ (7)

whence, $x^{\frac{1}{3}}=a$, or $-b;$
and $x=a^3$, or $-b^3;$
also, $y=-b^3$, or a^3.

NOTE.—Some may prefer to substitute v^3 for x and z^3 for y, and thus avoid fractional exponents.

10. $x^2y-xy^2=6;$ (1)
$x^3-y^3=19.$ (2)

Multiplying (1) by 3, $3x^2y-3xy^2=18;$ (3)
Subtracting (3) from (2), $x^3-3x^2y+3xy^2-y^3=1;$ (4)
Extracting the cube root, $x-y=1;$ (5)
Dividing (1) by (5), $xy=6;$ (6)
whence, $x=3$, or $-2;$
and $y=2$, or -3.

11. $x^2y+xy^2=20;$ (1)
$x^3+y^3=65.$ (2)

Multiplying (1) by 3, $3x^2y+3xy^2=60;$ (3)
Adding (2) and (3), $x^3+3x^2y+3xy^2+y^3=125;$ (4)
Extracting the cube root, $x+y=5;$ (5)
Dividing (1) by (5), $xy=4;$ (6)
whence, $x=4$, or $1;$
and $y=1$, or 4.

12. $2xy=2a^{\frac{1}{2}}b^{\frac{1}{2}};$ (1)
$x^4+y^4=a^2+b^2.$ (2)

Squaring (1), $4x^2y^2=4ab;$ (3)
Dividing (3) by 2, $2x^2y^2=2ab;$ (4)
Adding (2) and (4), $x^4+2x^2y^2+y^4=a^2+2ab+b^2;$ (5)
Extracting the square root, $x^2+y^2=\pm(a+b);$ (6)
Adding (1) and (6), $x^2+2xy+y^2=a+2a^{\frac{1}{2}}b^{\frac{1}{2}}+b;$ (7)
Extracting the square root, $x+y=\pm(\sqrt{a}+\sqrt{b});$ (8)
Subtracting (1) from (6), $x^2-2xy+y^2=a-2a^{\frac{1}{2}}b^{\frac{1}{2}}+b;$ (9)
Extracting the square root, $x-y=\pm(\sqrt{a}-\sqrt{b});$ (10)
Adding (8) and (10), $x=\sqrt{a}$, or $\sqrt{b};$
whence, $y=\sqrt{b}$, or $\sqrt{a}$.

13.

$$\sqrt[3]{x}-\sqrt[3]{y}=1; \qquad (1)$$
$$x-y=7. \qquad (2)$$

Dividing (2) by (1), $\sqrt[3]{x^2}+\sqrt[3]{xy}+\sqrt[3]{y^2}=7;$ (3)

Squaring (1), $\sqrt[3]{x^2}-2\sqrt[3]{xy}+\sqrt[3]{y^2}=1.$ (4)

Subtracting (4) from (3), $3\sqrt[3]{xy}=6;$

whence, $\sqrt[3]{xy}=2;$

and $xy=8;$ (5)

From (5), $x=\dfrac{8}{y};$

Substituting in (2), $\dfrac{8}{y}-y=7;$ (6)

Clearing of fractions, $8-y^2=7y;$

whence, $y=1, \text{ or } -8;$

and $x=8, \text{ or } -1.$

14.

$$xy=36; \qquad (1)$$
$$x-y=\sqrt{x}+\sqrt{y}. \qquad (2)$$

Dividing (2) by $\sqrt{x}+\sqrt{y}$, $\sqrt{x}-\sqrt{y}=1;$ (3)

whence, $\sqrt{x}=1+\sqrt{y};$ (4)

Extracting square root of (1), $\sqrt{xy}=\pm 6;$ (5)

Substituting from (4) in (5), $\sqrt{y}+y=\pm 6;$

whence, $y=4, \text{ or } 9;$

and $x=9, \text{ or } 4.$

15.

$$xy=6; \qquad (1)$$
$$x^2-3x+3y=10-y^2. \qquad (2)$$

Multiplying (1) by (2), $2xy=12;$ (3)

Subtracting (3) from (2), $x^2-2xy+y^2-3(x-y)=-2;$ (4)

Completing the square, $(x-y)^2-3(x-y)+\frac{9}{4}=\frac{1}{4};$ (5)

Extracting the square root, $x-y-\frac{3}{2}=\pm\frac{1}{2};$ (6)

whence, $x-y=1, \text{ or } 2;$ (7)

From (7), $x=y+1;$

Substituting in (1), $y^2+y=6;$

whence, $y=2, \text{ or } -3;$

and $x=3, \text{ or } -2.$

16.
$$x^3+y^3=\tfrac{7}{16}(x+y)^3; \quad (1)$$
$$xy=3. \quad (2)$$

Dividing (1) by $x+y$, $x^2-xy+y^2=\frac{7}{16}(x+y)^2$; (3)
Reducing, $9x^2-30xy+9y^2=0$; (4)
Multiplying (2) by 12, $12xy=36$; (5)
Adding (4) and (5), $9x^2-18xy+9y^2=36$; (6)
Extracting the square root, $3x-3y=\pm 6$;
Dividing by 3, $x-y=\pm 2$;
whence, $x=2+y$, or $y-2$; (7)
Substituting in (2), $2y+y^2=3$, or $y^2-2y=3$;
whence, $y=1$, or -3, or 3, or -1;
and $x=3$, or -1, or 1, or -3.

17.
$$x^{-2}+y^{-2}=\tfrac{13}{36}; \quad (1)$$
$$x^{-1}+y^{-1}=\tfrac{5}{6}. \quad (2)$$

Squaring (2), $x^{-2}+2x^{-1}y^{-1}+y^{-2}=\frac{25}{36}$; (3)
Subtracting (1) from (3), $2x^{-1}y^{-1}=\frac{12}{36}$; (4)
Subtracting (4) from (1), $x^{-2}-2x^{-1}y^{-1}+y^{-2}=\frac{1}{36}$; (5)
Extracting the square root, $x^{-1}-y^{-1}=\pm\frac{1}{6}$; (6)
Adding (2) and (6), $x^{-1}=\frac{1}{2}$, or $\frac{1}{3}$;
whence, $x=2$, or 3;
and $y=3$, or 2.

NOTE.—Negative exponents may be avoided by substitution.

18.
$$x^2+y^2=25; \quad (1)$$
$$x^4+y^4=337; \quad (2)$$

Squaring (1), $x^4+2x^2y^2+y^4=625$; (3)
Subtracting (2) from (3), $2x^2y^2=288$; (4)
Subtracting (4) from (2), $x^4-2x^2y^2+y^4=49$; (5)
Evolving, $x^2-y^2=\pm 7$; (6)
Adding (1) and (6), $x^2=16$, or 9;
whence, $x=\pm 4$, or ± 3,
and $y=\pm 3$, or ± 4.

19.
$$xy=6; \quad (1)$$
$$x^4+y^4=97. \quad (2)$$

Squaring (1), $x^2y^2=36$; (3)
Multiplying (3) by (2), $2x^2y^2=72$; (4)
Adding (4) and (2), $x^4+2x^2y^2+y^4=169$; (5)
Extracting the square root, $x^2+y^2=\pm 13$; (6)
Subtracting (4) from (2), $x^4-2x^2y^2+y^4=25$; (7)
Extracting the square root, $x^2-y^2=\pm 5$; (8)
Adding (6) and (8), $x^2=9$, or 4;
whence, $x=\pm 3$, or ± 2;
and $y=\pm 2$, or ± 3.

20.

$$x+y=5; \qquad (1)$$
$$x^4+y^4=257. \qquad (2)$$

Raising (1) to the 4th power, $x^4+4x^3y+6x^2y^2+4xy^3+y^4=625;$ (3)

Subtracting (2) from (3), $4x^3y+6x^2y^2+4xy^3=368;$ (4)

Sq. (1) and multiplying by $4xy$, $4x^3y+8x^2y^2+4xy^3=100xy.$ (5)

Subtracting (4) from (5), $2x^2y^2=100xy-368;$ (6)

Transposing and reducing, $x^2y^2-50xy=-184;$ (7)

Completing the square, $x^2y^2-50xy+625=441;$ (8)

Extracting the square root, $xy-25=\pm 21;$ (9)

whence, $xy=4$, or $46;$ (10)

From (10), $x=\dfrac{4}{y};$

Substituting in (1), $\dfrac{4}{y}+y=5;$

whence, $y=1$, or $4;$

and $x=4$, or 1.

PROBLEMS PRODUCING QUADRATICS WITH TWO UNKNOWN QUANTITIES.

1. Let $x=$ one number;

and $y=$ other number.

Then, $x+y=7;$ (1)

and $x^2+y^2=25.$ (2)

From (1), $x=7-y;$ (3)

Substituting in (2), $49-14y+y^2+y^2=25;$

Reducing, $y^2-7y=-12;$

whence, $y=3$, or $4;$

and $x=4$, or 3.

2. Let $x=$ first number;

and $y=$ second number.

Then, $x-y=2;$ (1)

and $x^2-y^2=20.$ (2)

Dividing (2) by (1), $x+y=10;$ (3)

Adding (1) and (3), $x=6;$

Subtracting (1) from (3), $y=4.$

3. Let $x^2 =$ the first part;
and $y^2 =$ the second part.

Then, $x^2+y^2=97;$ (1)
and $x+y=13.$ (2)

From (2), $x=13-y;$
Substituting in (1), $169-26y+y^2+y^2=97;$
Reducing, $y^2-13y=-36;$
whence, $y=4$, or $9;$
and $y^2=16$, or $81;$
then, $x^2=81$, or $16.$

4. Let $x^2 =$ the first number;
and $y^2 =$ the second number.

Then, $x^2-y^2=a;$ (1)
and $x-y=\frac{1}{2}\sqrt{2a};$ (2)
Dividing (1) by (2), $x+y=\sqrt{2a}.$ (3)

Adding (2) and (3), $x=\frac{3}{4}\sqrt{2a};$
whence $x^2=\frac{9}{8}a;$
and $y^2=\frac{a}{8}.$

5. Let $x =$ the first number;
$y =$ the second number.

Then, $xy=3(x+y);$ (1)
and $x^2+y^2=160;$ (2)
Multiplying (1) by 2, $2xy=6(x+y).$ (3)

Adding (2) and (3), $x^2+2xy+y^2=6(x+y)+160;$
Transposing, $(x+y)^2-6(x+y)=160;$ (4)
Completing the square, $(x+y)^2-6(x+y)+9=169;$ (5)
Extracting the square root, $x+y-3=\pm 13;$ (6)
whence, $x+y=16$, or $-10;$ (7)
Substituting in (3), $2xy=96;$ (8)
Subtracting (8) from (2), $x^2-2xy+y^2=64;$ (9)
Extracting the square root, $x-y=\pm 8;$ (10)
whence, $x=12$, or $4;$
and $y=4$, or $12.$

6. Let $x =$ the first part;
and $y =$ the second part.
Then, $x+y=10;$ (1)
and $x^3+y^3=280;$ (2)
Dividing (2) by (1), $x^2-xy+y^2=28;$ (3)
Squaring (1), $x^2+2xy+y^2=100.$ (4)
Subtracting (3) from (4), $3xy=72;$ (5)
Dividing (5) by 3, $xy=24;$ (6)
Subtracting (6) from (3), $x^2-2xy+y^2=4;$ (7)
Extracting the square root, $x-y=\pm 2;$ (8)
Adding (8) and (1), $x=6$, or $4;$
whence, $y=4$, or $6.$

This can also be solved by substituting the value of x in (2).

7. Let $x =$ the first number;
and $y =$ the second number.
Then, $x-y=3;$ (1)
and $x^3-y^3=117.$ (2)
Dividing (2) by (1), $x^2+xy+y^2=39;$ (3)
Squaring (1), $x^2-2xy+y^2=9.$ (4)
Subtracting (4) from (3), $3xy=30;$
whence, $xy=10;$ (5)
Adding (3) and (5), $x^2+2xy+y^2=49;$ (6)
Extracting the square root, $x+y=\pm 7;$ (7)
Adding (1) and (7), $x=5$, or $-2;$
whence, $y=2$, or $-5.$

8. Let $x =$ the first number;
and $y =$ the second number.
Then, $xy=6(x-y);$ (1)
and $x^2+y^2=13.$ (2)
Multiplying (1) by 2, $2xy=12(x-y);$ (3)
Subtracting (3) from (2,) $x^2-2xy+y^2=13-12(x-y);$ (4)
Transposing, $(x-y)^2+12(x-y)=13;$ (5)
Completing the sq. $(x-y)^2+12(x-y)+36=49;$ (6)
Extracting the square root, $x-y+6=\pm 7;$
whence, $x-y=1$, or $-13;$ (7)
and $x=1+y;$
Substituting in (2), $1+2y+y^2+y^2=13;$
whence, $y=2$, or $-3;$
and $x=3$, or $-2.$

9. Let $x =$ first number;

and $y =$ second number.

Then, $x+y=a;$ (1)

and $x^3+y^3=4a^3.$ (2)

From (1), $x=a-y;$ (3)

Substituting in (2), $a^3-3a^2y+3ay^2-y^3+y^3=4a^3;$

Reducing, $3ay^2-3a^2y=3a^3;$

Dividing by $3a$, $y^2-ay=a^2;$ (4)

Completing the square, $y^2-ay+\frac{a^2}{4}=\frac{5a^2}{4};$

Extracting the root, $y-\frac{a}{2}=\pm\frac{a}{2}\sqrt{5};$

whence, $y=\frac{a}{2}(1\pm\sqrt{5});$

and $x=\frac{a}{2}(1\mp\sqrt{5})$

10. Let $x =$ number of days it takes A;

and $y =$ number of days it takes B.

Then, $y-x=10;$ (1)

and $\frac{1}{x}+\frac{1}{y}=\frac{1}{12}.$ (2)

Clearing (2) of fractions, $12y+12x=xy;$ (3)

From (1), $y=10+x;$ (4)

Substituting in (3), $120+12x+12x=10x+x^2;$ (5)

Reducing, $x^2-14x=120;$

whence, $x=20$, or $-6;$

and $y=30$, or 4.

11. Let $x =$ the side of first square;

and $y =$ the side of second square.

Then, $y-x=5;$ (1)

and $x^2+y^2=1025.$ (2)

From (1), $y=x+5;$ (3)

Substituting in (2), $x^2+x^2+10x+25=1025;$ (4)

whence, $x=20$, or $-25;$

and $y=25$, or -20.

12. Let x = the price of one calf;
and y = the price of one sheep.

Then, $7x+12y=50;$ (1)

and $\frac{10}{x}-\frac{6}{y}=3.$ (2)

Clearing (2) of fractions, $10y-6x=3xy;$ (3)

From (1), $x=\frac{50-12y}{7};$ (4)

Substituting in (3), $10y-\frac{300-72y}{7}=\frac{150y-36y^2}{7};$

Reducing, $36y^2-8y=300;$
whence, $y=3, \text{ or } -\frac{25}{9};$
and $x=2.$

13. Let x = first number;
and y = second number.

Then, $x-y+x^2-y^2=6;$ (1)
$x+y+x^2+y^2=18.$ (2)

Adding (1) and (2), $2x+2x^2=24;$ (3)
whence, $x=3, \text{ or } -4;$
and $y=2, \text{ or } -3.$

Many of these examples can be solved by using one unknown quantity, by the methods previously given. We have endeavored to give as great a variety of solutions as possible.

14. Let x = the original number of young men;
and y = the number left;
then, $\frac{70}{x}$ = each one's share by first condition;
and $\frac{70}{y}$ = each one's share by second condition.

Then, $x-y=4;$ (1)

and $\frac{70}{y}-\frac{70}{x}=2.$ (2)

Clearing (2) of fractions, $70x-70y=2xy;$ (3)
Substituting (1) in (3), $2xy=280;$ (4)
From (1), $x=y+4;$
Substituting in (4), $2y^2+8y=280;$
whence, $y=10;$
and $x=14.$

15. Let $x=$ the number of yards of cloth;

and $x+2=$ the number of yards of silk;

then, $y=$ the price of one yard of cloth;

and $y-1=$ the price of one yard of silk.

Then, $xy=24;$ (1)

and $xy+2y-x-2=24.$ (2)

Subtracting (1) from (2), $2y-x=2;$ (3)

From (1), $x=\frac{24}{y};$

Substituting in (3), $2y-\frac{24}{y}=2;$ (4)

Clearing of fractions, $2y^2-24=2y;$ (5)

whence, $y=4;$

and $x=6;$

and $x+2=8$

16. Let $x=$ the number of miles A runs in one hour;

and $y=$ the number of miles B runs in one hour;

then, $\frac{1}{x}=$ the time A runs one mile;

and $\frac{1}{y}=$ the time B runs one mile.

Then, $x-y=\frac{1}{2};$ (1)

and $\frac{4}{y}-\frac{4}{x}=\frac{2}{60}=\frac{1}{30};$ (2)

Clearing (2) of fractions, $120x-120y=xy;$ (3)

Substituting (1) in (3), $xy=60;$

whence, $x=\frac{60}{y};$ (4)

Substituting in (1), $\frac{60}{y}-y=\frac{1}{2};$

Clearing of fractions, $120-2y^2=y;$ (5)

whence, $y=7\frac{1}{2}$, or $-8;$

and $x=8$, or $-7\frac{1}{2}$.

17. Let $x =$ the length of the first rectangle;
and $y =$ the breadth of the first rectangle;
then, $x - 8 =$ the length of the second rectangle;
and $y + 10 =$ the breadth of the second rectangle.

Then, $$xy = 300; \quad (1)$$
and $$xy - 8y + 10x - 80 = 300. \quad (2)$$
Subtracting (1) from (2), $$10x - 8y = 80; \quad (3)$$
From (1), $$x = \frac{300}{y};$$
Substituting in (3), $$\frac{3000}{y} - 8y = 80; \quad (4)$$
Clearing of fractions and dividing by 8, $$375 - y^2 = 10y; \quad (5)$$
whence, $$y = 15;$$
and $$x = 20.$$

18. Let $x =$ the number of yards of finer;
and $x + 10 =$ the number of yards of coarser;
then, $y =$ the price of one yard of finer;
and $y - 1 =$ the price of one yard of coarser.

Then, $$xy = 90; \quad (1)$$
and $$xy - x + 10y - 10 = 80. \quad (2)$$
Subtracting (2) from (1), $$x - 10y = 0;$$
whence, $$x = 10y; \quad (3)$$
Substituting in (1), $$10y^2 = 90;$$
whence, $$y = \pm 3;$$
and $$x = 30;$$
and $$x + 10 = 40.$$

19. Let $x =$ the length of field;
and $y =$ the breadth of field.

Then, $$xy = 2275; \quad (1)$$
and $$(x-5)(y-5) = 1800. \quad (2)$$
Expanding (2), $$xy - 5x - 5y + 25 = 1800; \quad (3)$$
Subtracting (3) from (1), $$5x + 5y = 500; \quad (4)$$
Dividing by 5, $$x + y = 100; \quad (5)$$
whence, $$x = 100 - y;$$
Substituting in (1), $$100y - y^2 = 2275;$$
whence, $$y = 35;$$
and $$x = 65.$$

20. Let $x=$ the tens' digit;
and $y=$ the units' digit;
then, $10x+y=$ the number.

Then, $x^2+y^2=10x+y+xy;$ (1)
and $10x+y+36=10y+x.$ (2)

Reducing (2), $y-x=4;$
whence, $y=4+x;$ (3)
Substituting in (1), $x^2+16+8x+x^2=10x+4+x+4x+x^2;$
Reducing, $x^2-7x=-12;$
whence, $x=4$, or $3;$
and $y=8$, or $7.$

21. Let $x=$ the number of yards in side of first;
and $y=$ the number of yards in side of second;
then, $x^3y=$ the value of first stack in shillings;
and $xy^3=$ the value of second stack in shillings.

Then, $x^3y+xy^3=41\times 20=820;$ (1)
$x^2-y^2=9.$ (2)

Dividing (1) by xy, $x^2+y^2=\frac{820}{xy};$ (3)
Squaring (2), $x^4-2x^2y^2+y^4=81;$ (4)
Squaring (3), $x^4+2x^2y^2+y^4=\frac{(820)^2}{x^2y^2};$ (5)
Subtracting (4) from (5), $4x^2y^2=\frac{(820)^2}{x^2y^2}-81;$ (6)
Clearing (6) of fractions, $4x^4y^4=(820)^2-81x^2y^2;$ (7)
Reducing, $x^4y^4+\frac{81}{4}x^2y^2=168100;$ (8)
whence, $x^2y^2=400;$
and $xy=\pm 20;$
Substituting in (3), $x^2+y^2=41;$ (9)
Adding (9) and (2), $x^2=25;$
whence, $x=\pm 5;$
and $y=\pm 4;$
then, $\frac{x^3y}{20}=25;$
and $\frac{xy^3}{20}=16.$

22. Let x = the length of first trench;
and y = the length of second trench.

Then, $y - x = 6;$ (1)
and $x^2 + y^2 = 356.$ (2)

From (1), $y = 6 + x;$ (3)
Substituting in (2), $x^2 + 36 + 12x + x^2 = 356;$ (4)
whence, $x = 10;$
and $y = 16.$

23. Let x = the first number;
and y = the second number.

Then, $x + y = xy;$ (1)
and $x + y = x^2 - y^2.$ (2)

Dividing (2) by $x+y$, $x - y = 1;$ (3)
From (3), $x = 1 + y;$
Substituting in (1), $1 + y + y = y + y^2;$ (4)
whence, $y = \frac{1}{2} \pm \frac{1}{2}\sqrt{5};$
and $x = \frac{3}{2} \pm \frac{1}{2}\sqrt{5}.$

24. Let x = A's principal;
and y = the gain on 1 dollar in one month,

Then, $x + 12xy = 26;$ (1)
and $12xy + 30 \times 16y = 18.$ (2)

Subtracting (2) from (1), $x - 480y = 8;$ (3)
whence, $x = 8 + 480y;$ (4)
Substituting in (1), $8 + 480y + 96y + 5760y^2 = 26;$ (5)
Reducing, $y^2 + \frac{1}{10}y = \frac{1}{320};$
Completing the square, $y^2 + \frac{1}{10}y + \frac{1}{400} = \frac{9}{1600};$
Evolving, $y + \frac{1}{20} = \pm \frac{3}{40};$
whence, $y = \frac{1}{40};$
and $x = 20.$

25. Let x = circumference of fore wheel;
and y = circumference of hind wheel;

then, $\frac{60}{x}$ = number of revolutions fore wheel makes in 60 yards;

and $\frac{60}{y}$ = number of revolutions hind wheel makes in 60 yards.

Then, $\frac{60}{x} - \frac{60}{y} = 5;$ (1)

and $\frac{60}{x+1} - \frac{60}{y+1} = 3.$ (2)

Clearing (1) of fractions, $60y - 60x = 5xy;$ (3)

Clearing (2) of fractions, $60y + 60 - 60x - 60 = 3xy + 3y + 3x + 3;$ (4)

Reducing (4), $19y - 21x = xy + 1;$ (5)

Dividing (3) by 5, $12y - 12x = xy.$ (6)

Subtracting (6) from (5), $7y - 9x = 1;$

whence, $y = \frac{1+9x}{7};$ (7)

Substituting in (6), $\frac{12+108x}{7} - 12x = \frac{x+9x^2}{7};$ (8)

Reducing, $12 + 23x = 9x^2;$

whence, $x = 3;$

and $y = 4.$

26. Let $x =$ the number of bushels of wheat;

then, $x + 16 =$ the number of bushels of barley;

let $y =$ the price of one bushel of wheat;

then, $y - 1\frac{1}{2} =$ the price of one bushel of barley.

Then, $xy = 144;$ (1)

and $(x+16)(y - 1\frac{1}{2}) = 144.$ (2)

Expanding, $xy + 16y - 1\frac{1}{2}x - 24 = 144;$ (3)

Subtracting (1) from (3), $16y - \frac{3x}{2} = 24;$ (4)

From (1), $y = \frac{144}{x};$

Substituting in (4), $16\left(\frac{144}{x}\right) - \frac{3x}{2} = 24;$ (5)

Clearing of fractions, $1536 - x^2 = 16x;$

whence, $x = 32;$

and $x + 16 = 48.$

27. Let x = the number of miles per hour A runs;
and y = the number of miles per hour B runs;
then, $\frac{2}{x}$ = time A runs the first race;
and $\frac{2}{y}$ = time B runs the first race.

Then, $$\frac{2}{x}-\frac{2}{y}=\tfrac{2}{60}=\tfrac{1}{30}; \tag{1}$$
and $$\frac{2}{y-2}-\frac{2}{x+2}=\tfrac{2}{60}=\tfrac{1}{30}; \tag{2}$$

Clearing (1) of fractions, $$60y-60x=xy; \tag{3}$$
Clearing (2) of fractions, $$60x+120-60y+120=xy-2x+2y-4; \tag{4}$$
Reducing, $$62x-62y=xy-244; \tag{5}$$
Subtracting (5) from (3), $$122y-122x=244; \tag{6}$$
whence, $$y=2+x;$$
Substituting in (3), $$120+60x-60x=2x+x^2; \tag{7}$$
whence, $$x=10;$$
and $$y=12.$$

Art. 305. (page 231.)

3. $$x^2-(7-3)x=-(7\times-3);$$
$$x^2-4x=21.$$

6. $$x^2-(2+\sqrt{3}+2-\sqrt{3})x=-(2+\sqrt{3})(2-\sqrt{3});$$
$$x^2-4x=-1.$$

9. $$x^2-(a+b\sqrt{c}+a-b\sqrt{c})x=-(a+b\sqrt{c})(a-b\sqrt{c});$$
$$x^2-2ax=b^2c-a^2$$

Art. 306. (page 232.)

5. $$(x-a)(x+a)=0;$$
$$x^2-a^2=0;$$
$$x^2=a^2.$$

6. $$\left(x-\frac{m}{2}\right)\left(x+\frac{n}{2}\right)=0;$$
$$x^2+\tfrac{1}{2}(n-m)x-\frac{mn}{4}=0;$$
$$x^2+\tfrac{1}{2}(n-m)x=\tfrac{1}{4}mn.$$

7. $$(x-a-2\sqrt{n})(x-a+2\sqrt{n})=0;$$
$$x^2-2ax+a^2-4n=0;$$
$$x^2-2ax=4n-a^2.$$

Art. 308. (page 235.)

1. Let x = the number. Then, $x^2+4x=0$; comparing with general equation, $2p=4$; and $q=0$; substituting in the root, $x=0$, or -4.

This example illustrates the first supposition.

2. Let x = the number. Then, $x^2+6x=-9$; comparing with general equation, $2p=6$; and $q=-9$; substituting in the root, $x=-3$.

This example illustrates the third supposition.

3. Let x = the one part; and $8-x$ = the other part.

Then, $x(8-x)=20$; expanding, $x^2-8x=-20$; comparing with general equation, $2p=-8$; and $q=-20$; whence, $x=4\pm2\sqrt{-1}$.

This root becomes imaginary, because q, or 20, is negative and greater than p^2, or 16.

Art. 311. (page 237.)

1. Let x = the one part; and $12-x$ = the other part.

Then, $x(12-x)=40$; whence, $x^2-12x=-40$; and $x=6\pm2\sqrt{-1}$.

2. Let x = the length; $10-x$ = the breadth.

Then, $10x-x^2=40$; completing the square, $x^2-10x+25=-15$; whence, $x=5\pm\sqrt{-15}$; and $10-x=5\mp\sqrt{-15}$.

These problems give an imaginary result because, when reduced to the regular form, the known term is negative and numerically greater than the square of half the coefficient of x. In the first the square of half the sum of the required numbers, which is 36, is less than their product, which is 40. In the second, 25, the square of half the sum of the sides is less than 40, their product.

RATIO AND PROPORTION.

Art. 320. (page 240.)

7. $\frac{a}{b}=\frac{8}{15}$; multiplying by $\frac{5}{4}$, $\frac{5a}{4b}=\frac{5}{4}\times\frac{8}{15}=\frac{2}{3}$.

8. $\frac{3a}{2b}=\frac{2}{3}$; dividing by $\frac{3}{2}$, $\frac{a}{b}=\frac{2}{3}\times\frac{2}{3}=\frac{4}{9}$.

9. $\frac{2m}{5n}=\frac{4}{5}$; dividing by $\frac{2}{5}$, $\frac{m}{n}=2$· multiplying by $\frac{5}{2}$, $\frac{5m}{2n}=2\times\frac{5}{2}=5$.

10. $\frac{a}{c}=\frac{4}{5}$; $a=\frac{4c}{5}$; $a+c=\frac{9c}{5}$; $a-c=\frac{-c}{5}$ $\therefore \frac{a+c}{a-c}=\frac{9c}{5}\div\frac{-c}{5}=-9.$

11. $(a+b)(a-b)=a^2-b^2.$

Art. 321. (page 241.)

4. $\left\{\begin{matrix} x:8 \\ 6:9 \end{matrix}\right\}=\frac{8}{15}$; $\frac{6x}{72}=\frac{8}{15}$; $x=6\frac{2}{5}.$

5. $\left\{\begin{matrix} 9:12 \\ a:18 \end{matrix}\right\}=\frac{1}{4}$; $\frac{9a}{216}=\frac{1}{4}$; $a=6.$

6. $\left\{\begin{matrix} 16:15 \\ 21:c \end{matrix}\right\}=7$; $\frac{112}{5c}=7$; $c=3\frac{1}{5}.$

7. $2:3\frac{1}{3}=\left\{\begin{matrix} 16:15 \\ 18:c \end{matrix}\right\}$; $\frac{3}{5}=\frac{96}{5c}$; $c=32.$

8. $x^2:1=27:x$; $x^2=\frac{27}{x}$; $x=3.$

9. $a-x:b-x=a^2:b^2$; $\frac{a-x}{b-x}=\frac{a^2}{b^2}$; $ab^2-b^2x=a^2b-a^2x$; $(a^2-b^2)x$

$=a^2b-ab^2$; $x=\frac{a^2b-ab^2}{a^2-b^2}=\frac{ab}{a+b}.$

10. $x^2:a^2=\left\{\begin{matrix} x^2:a^2 \\ b^2:x \end{matrix}\right\}$; $\frac{x^2}{a^2}=\frac{b^2x^2}{a^2x}$; $x=b^2.$

Art. 336. (page 247.)

1. Let $a:b::c:d;$

then, $\frac{a}{b}=\frac{c}{d};$

Multiplying by m, $\frac{am}{b}=\frac{cm}{d};$

Dividing by n, $\frac{am}{bn}=\frac{cm}{dn};$

whence, $am:bn::cm:dn.$

2. Let $a:b::c:d;$

then, $\frac{a}{b}=\frac{c}{d};$

Multiplying by $\frac{a}{b}$, $\frac{a^2}{b^2}=\frac{ac}{bd};$

whence, $a^2:b^2::ac:bd.$

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3. Let $a : b :: c : d$;

then, $\frac{a}{b} = \frac{c}{d}$;

by Theorem XI., $\frac{a^2}{ab} = \frac{c^2}{cd}$;

hence, $a^2 : ab :: c^2 : cd$;

but by Theorem V., $a^2 : c^2 :: ab : cd$;

4. Let $a : b :: c : d$;

then by Theorem VI., $b : a :: d : c$;

whence, $\frac{b}{a} = \frac{d}{c}$;

Subtracting from 1, $1 - \frac{b}{a} = 1 - \frac{d}{c}$;

Reducing, $\frac{a-b}{a} = \frac{c-d}{c}$;

whence, $a-b : a :: c-d : c$,

and by Theorem VI., $a : a-b :: c : c-d$.

5. Let $a : b :: c : d$;

by Theorem VII., $a+b : b :: c+d : d$;

and by Theorem V., $a+b : c+d :: b : d$.

By Theorem VIII., $a-b : b :: c-d : d$;

and by Theorem V., $a-b : c-d :: b : d$;

then by Theorem XII., $a+b : c+d :: a-b : c-d$.

6. Let $a : b :: c : d$;

then, $\frac{a}{b} = \frac{c}{d}$;

Multiplying by $\frac{n}{m}$, $\frac{na}{mb} = \frac{nc}{md}$,

Adding 1, $\frac{na+mb}{mb} = \frac{nc+ma}{md}$; (1)

also, $\frac{a}{mb} = \frac{c}{md}$; (2)

Dividing (1) by (2), $\frac{na+mb}{a} = \frac{nc+md}{c}$;

whence, $na+mb : a :: nc+md : c$,

and by Theorem VI., $a : na+mb :: c : nc+md$.

7. Let $$a : b :: b : c;$$
then, $$\frac{a}{b}=\frac{b}{c};$$
Multiplying by $\frac{a}{b}$, $$\frac{a^2}{b^2}=\frac{a}{c};$$
whence, $$a^2 : b^2 :: a : c.$$

8. Let $$a : b :: b : c;$$
then, $$\frac{a}{b}=\frac{b}{c};$$
Multiplying by $\frac{b}{c}$, $$\frac{a}{c}=\frac{b^2}{c^2};$$
whence, $$a : c :: b^2 : c^2.$$

9. Let $$a : b :: b : c;$$
then, $$\frac{a}{b}=\frac{b}{c};$$
Squaring, $$\frac{a^2}{b^2}=\frac{b^2}{c^2};$$
Subtracting 1, $$\frac{a^2-b^2}{b^2}=\frac{b^2-c^2}{c^2}.$$
But $$b^2=ac;$$
Substituting, $$\frac{a^2-b^2}{ac}=\frac{b^2-c^2}{c^2};$$
or, $$\frac{a^2-b^2}{a}=\frac{b^2-c^2}{c};$$
whence, $$a^2-b^2 : a :: b^2-c^2 : c.$$

10. Let $$a : b :: b : c;$$
then, $$\frac{a}{b}=\frac{b}{c};$$
Squaring, $$\frac{a^2}{b^2}=\frac{b^2}{c^2};$$
Adding 1, $$\frac{a^2+b^2}{b^2}=\frac{b^2+c^2}{c^2}; \qquad (1)$$
Subtracting 1, $$\frac{a^2-b^2}{b^2}=\frac{b^2-c^2}{c^2}; \qquad (2)$$
Dividing (1) by (2), $$\frac{a^2+b^2}{a^2-b^2}=\frac{b^2+c^2}{b^2-c^2}.$$
But $$b^2=ac;$$
hence, $$\frac{a^2+b^2}{a^2-b^2}=\frac{ac+c^2}{ac-c^2};$$
and $$a^2+b^2 : a^2-b^2 :: a+c : a-c.$$

11. Let $a : b :: b : c$

then from the last Theorem, $\dfrac{a^2+b^2}{b^2} = \dfrac{b^2+c^2}{c^2}$;

Multiplying by $(b^2+c^2)b^2$,

$$(a^2+b^2)(b^2+c^2) = \frac{(b^2+c^2)^2b^2}{c^2}.$$

But $b^2+c^2 = ac+c^2 = (a+c)c$;

hence, $\dfrac{(b^2+c^2)^2b^2}{c^2} = \dfrac{(a+c)^2c^2b^2}{c^2} = (a+c)^2b^2 = (ab+bc^2)$

therefore, $(a^2+b^2)(b^2+c^2) = (ab+bc)^2$.

Art. 337. (page 249.)

4. Let $2x$ = the first number; and $3x$ = the second number.

Then, $2x+3 : 3x+3 :: 5 : 7$; by Theorem I., $14x+21 = 15x+15$; whence, $x=6$; and $2x=12$; and $3x=18$.

5. Let $4x$ = the first number; and $5x$ = the second number.

Then, $4x-6 : 5x-6 :: 3 : 4$; by Theorem I., $16x-24 = 15x-18$; whence, $x=6$; and $4x=24$; and $5x=30$.

6. Let $3x$ = the first number; and $5x$ = the second number.

Then, $3x-2 : 5x+5 :: 2 : 5$; by Theorem I., $15x-10 = 10x+10$; then, $5x=20$; and $3x=12$.

7. Let x = the number. Then, $\dfrac{5+x}{3+x} = \frac{3}{4}\left(\dfrac{5-x}{3-x}\right)$; clearing of fractions, $60-8x-4x^2 = 45+6x-3x^2$; transposing and uniting, $x^2+14x=15$; whence, $x=1$.

8. Let $2x$ = the first number; and $3x$ = the second number.

Then, $x : 5x^2 :: 1 : 25$; whence, $5x^2 = 25x$; and $x=5$; then, $2x=10$; and $3x=15$.

9. Let $3x$ = the first number; and $4x$ = the second number.

Then, $7x : 25x^2 :: 7 : 50$; whence, $175x^2 = 350x$; and $x=2$; then, $3x=6$; and $4x=8$.

10. Let $5x$ = the first number; and $6x$ = the second number.

Then, $11x : 11x^2 :: 1 : 7$; whence, $11x^2 = 77x$; and $x=7$; then, $5x=35$; and $6x=42$.

11. Let $x =$ the first number;
and $y =$ the second number.

Then, $x+y = 10;$ (1)
and $x^2+y^2 : x^2-y^2 :: 13 : 5.$ (2)
Then by composition and division, $x^2 : y^2 :: 9 : 4;$
whence, $4x^2 = 9y^2;$
and $2x = 3y;$ (3)
Substituting in (1), $\frac{3y}{2}+y = 10;$
whence, $y = 4;$
and $x = 6.$

12. Let $x =$ the less number;
and $x+6 =$ the greater number.

Then, $x^2+6x : 2x^2+12x+36 :: 2 : 5;$
whence, $5x^2+30x = 4x^2+24x+72;$
Transposing and uniting, $x^2+6x = 72;$
whence, $x = 6;$
and $x+6 = 12.$

13. Let $3x =$ one number;
and $2x =$ the other number.

Then, $3x+6 : 2x-6 :: 3 : 1;$
whence, $3x+6 = 6x-18;$
then, $3x = 24;$
and $2x = 16.$

14. Let $x =$ one number;
and $y =$ the other number.

Then, $xy = 12;$ (1)
and $x^3-y^3 : x^3+y^3 :: 13 : 14.$ (2)
By composition and division, $x^3 : y^3 :: 27 : 1;$
whence, $x^3 = 27y^3;$
and $x = 3y;$
Substituting in (1), $3y^2 = 12;$
whence, $y = 2;$
and $x = 6.$

15. Let $x =$ the first number;
and $y =$ the third number.

Then, $x+y = 125;$ (1)
and $x : 60 :: 60 : y.$ (2)
From (2), $xy = 3600;$ (3)
then, $x = 45;$
and $y = 80.$

16. Let $6x$ = the number of gallons of milk;
and $7x$ = the number of gallons of the mixture.

Then, $7x : 7x-8 :: 7 : 5$;
whence, $35x = 49x - 56$;
and $x = 4$;
and $6x = 24$.

PROGRESSIONS.

Art. 347. (page 253.)

9. $l = a + (n-1)d$.
Substituting, $l = a + (30-1)2a$;
Reducing, $l = a + 58a = 59a$.

10. $l = a + (n-1)d$.
Substituting, $l = 2 + (n-1)2$;
Reducing, $l = 2 + 2n - 2 = 2n$.

11. $l = a + (n-1)d$.
Substituting, $l = 2b + (n-1)2b$;
Reducing, $l = 2b + 2bn - 2b = 2bn$.

12. $l = a + (n-1)d$.
Substituting, $l = 1 + (n-1)2$;
Reducing, $l = 1 + 2n - 2 = 2n - 1$.

13. $l = a + (n-1)d$.
Substituting, $l = 2 + (n-1)\frac{1}{3}$;
Reducing, $l = 2 + \frac{n}{3} - \frac{1}{3} = \frac{1}{3}(n+5)$.

14. $l = a + (n-1)d$.
Substituting, $l = 16\frac{1}{12} + (20-1)32\frac{1}{6}$;
Reducing, $l = 16\frac{1}{12} + 19 \times 32\frac{1}{6} = 627\frac{1}{4}$.

15. $l = a + (n-1)d$.
Substituting, $l = n + (t-1)2n$;
Reducing, $l = n + 2tn - 2n = (2t-1)n$.

Art. 348. (page 254.)

11. $S=\frac{n}{2}(a+l)$.

From Case I., $l=.2+(17-1).05=1$;

Substituting, $S=\frac{17}{2}(.2+1)=10.2$.

2. $S=\frac{n}{2}(a+l)$.

From Case I., $l=1+(n-1)2=2n-1$;

Substituting. $S=\frac{n}{2}(1+2n-1)=n^2$.

13. $S=\frac{n}{2}(a+l)$.

From Case I., $l=2+(n-1)2=2n$;

Substituting, $S=\frac{n}{2}(2+2n)=n^2+n$.

14. $S=\frac{n}{2}(a+l)$.

From Case I., $l=a+(n-1)2a=2an-a$;

Substituting, $S=\frac{n}{2}(a+2an-a)=an^2$.

15. $S=\frac{n}{2}(a+l)$.

From Case I., $l=a-5b+(6-1)2b$;
whence, $l=a+5b$;
Substituting, $S=\frac{6}{2}(a-5b+a+5b)$;
whence, $S=6a$.

16 $S=\frac{n}{2}(a+l)$.

From Case I., Example 14, $l=627\frac{1}{4}$;

Substituting, $S=\frac{20}{2}(16\frac{1}{12}+627\frac{1}{4})$;

whence, $S=6433\frac{1}{3}$.

17. $S=\frac{n}{2}(a+l)$.

From Case I., $l=n+(t-1)2n=2tn-n$;

Substituting, $S=\frac{t}{2}(n+2tn-n)=t^2n$.

Art. 352. (page 255.)

2. Given, $l=a+(n-1)d$. Transposing, $a=l-(n-1)d$.

3. Given, $l=a+(n-1)d$. Transposing, $n=\dfrac{l-a+d}{d}$.

4. Given, $l=a+(n-1)d$. Transposing, $n=\dfrac{l-a+d}{d}$; substituting, $n=\dfrac{34-90-4}{-4}=15$.

5. Given, $n=\dfrac{l-a+d}{d}$. Substituting, $n=\dfrac{35-2+3}{3}=12$. Substituting, $n=\dfrac{3-29-2}{-2}=14$.

6. Given, $a=l-(n-1)d$. Substituting, $a=2n-(n-1)2=2$.

7. Given, $d=\dfrac{l-a}{n-1}$. Substituting, $d=\dfrac{2n-1-1}{n-1}=2$; then, 1, 3, 5, 7 are the required terms.

Art. 353. (page 256.)

2. $S=\dfrac{n}{2}(a+l)$.

Transposing, $a+l=\dfrac{2S}{n}$;

and $a=\dfrac{2S}{n}-l$.

3. $S=(a+l)\dfrac{n}{2}$.

Transposing, $a+l=\dfrac{2S}{n}$;

and $l=\dfrac{2S}{n}-a$.

4. $n=\dfrac{2S}{a+l}$.

Substituting, $n=\dfrac{444}{35+2}=12$.

5. $a=\dfrac{2S}{n}-l$.

Substituting, $a=\dfrac{360}{12}-27=3$.

6. $l=\dfrac{2S}{n}-a$.

Substituting, $l=\dfrac{2n^2}{n}-1=2n-1$.

7. $l=\dfrac{2S}{n}-$

Substituting, $l=\dfrac{2n^2}{n}-1=2n-1$;

Making $n=1$, $l=2-1=1$;
Making $n=2$, $l=4-1=3$;
Making $n=3$, $l=6-1=5$;
Making $n=4$, $l=8-1=7$.

8. $$l=\frac{2S}{n}-a.$$

Substituting, $$l=\frac{2an^2}{n}-a=a(2n-1)$$

Art. 356. (page 258.)

5. Given, $d=\frac{l-a}{n-1}$. Substituting, $d=\frac{b-a}{3-1}=\frac{b-a}{2}$; then, $a+\frac{b-a}{2}=\frac{a+b}{2}$.

6. Given, $d=\frac{l-a}{n-1}$. Substituting, $d=\frac{b-a}{4-1}=\frac{b-a}{3}$; then, $a+\frac{b-a}{3}=\frac{2a+b}{3}$; and $\frac{2a+b}{3}+\frac{b-a}{3}=\frac{a+2b}{3}$.

7. Given, $d=\frac{l-a}{n-1}$. Substituting, $d=\frac{b-a}{5-1}=\frac{b-a}{4}$; then, $a+\frac{b-a}{4}=\frac{3a+b}{4}$; $\frac{3a+b}{4}+\frac{b-a}{4}=\frac{a+b}{2}$; $\frac{a+b}{2}+\frac{b-a}{4}=\frac{a+3b}{4}$.

8. Given, $d=19-5=14$; then, $l=19+14=33$.

Art. 362. (page 260.)

3. Let $n=1$; then, $\frac{1}{3}(1+5)=2$, first term.
Let $n=2$; then, $\frac{1}{3}(2+5)=2\frac{1}{3}$, second term.
Let $n=3$; then, $\frac{1}{3}(3+5)=2\frac{2}{3}$, third term, etc.

4. Let $n=1$; then, $\frac{1}{6}(3-1)=\frac{1}{3}$=first term.
Let $n=2$; then, $\frac{1}{6}(6-1)=\frac{5}{6}$=second term.
Subtracting 1st term from 2d, $\frac{5}{6}-\frac{1}{3}=\frac{1}{2}$, common difference.
Substituting in formula (2), $S=\frac{n}{2}\left\{\frac{1}{6}(3n-1)+\frac{1}{3}\right\}=\frac{n}{12}(3n+1)$.

5. Let $n=1$; then, $(2n-1)a=a$, first term.
Then, $$S=\frac{n}{2}(a+(2n-1)a)=an^2.$$

6. Let $x-y$=first term; x=second term; and $x+y$=third term.
Then, $3x=12$, (1); and $x^3-xy^2=48$, (2).
From (1), $x=4$; substituting in (2), $64-4y^2=48$; whence, $y=2$; and $x-y=2$; and $x+y=6$.

7. Let $$x-y=\text{first term}$$
$$x=\text{second term};$$
$$x+y=\text{third term.}$$

Then, $$3x=18; \quad (1)$$
and $$x^2-xy=24. \quad (2)$$
From (1), $$x=6;$$
Substituting in (2), $$36-6y=24;$$
whence, $$y=2;$$
and $$x-y=4;$$
and $$x+y=8.$$

8. Let $x-y$, x and $x+y$ be the terms of the progression.

Then, $$3x=15; \quad (1)$$
and $$x^3-3x^2y+3xy^2-y^3+x^3+x^3+3x^2y+3xy^2+y^3=645. \quad (2)$$
Reducing (2), $$3x^3+6xy^2=645; \quad (3)$$
From (1), $$x=5;$$
Substituting in (3), $$375+30y^2=645;$$
whence, $$y=3;$$
then, $$x-y=2;$$
and $$x+y=8.$$

9. Let $x-3y$, $x-y$, $x+y$ and $x+3y$ be the terms.

Then, $$2x=8; \quad (1)$$
and $$x^2-y^2=15. \quad (2)$$
From (1), $$x=4;$$
Substituting in (2), $$16-y^2=15;$$
whence, $$y=1;$$
Then, $$x-3y=1,\ x-y=3,\ x+y=5 \text{ and } x+3y=7.$$

10. Let $x-3y$, $x-y$, $x+y$ and $x+3y$ be the required terms.

Then, $$x^2-9y^2=22; \quad (1)$$
and $$x^2-y^2=40. \quad (2)$$
Subtracting (1) from (2), $$8y^2=18;$$
whence, $$y=\tfrac{3}{2};$$
Substituting in (2), $$x^2-\tfrac{9}{4}=40;$$
whence, $$x=\tfrac{13}{2};$$
Then, $$x-3y=2,\ x-y=5,\ x+y=8 \text{ and } x+3y=11.$$

11. Let $x-3y$, $x-y$, $x+y$ and $x+3y$ be the required terms.

Then, $\quad 4x=30; \quad (1)$

and $\quad x^3-3x^2y+3xy^2-y^3+x^3+3x^2y+3xy^2+y^3=945. \quad (2)$

Reducing (2), $\quad 2x^3+6xy^2=945; \quad (3)$

From (1), $\quad x=\frac{15}{2};$

Substituting in (3), $\quad \frac{3375}{4}+45y^2=945;$

whence, $\quad y=\frac{3}{2}.$

Therefore the terms are 3, 6, 9 and 12.

12. Let $\quad x-3y$, $x-y$, $x+y$, $x+3y$ be the numbers.

Then, $\quad 4x=22; \quad (1)$

and $\quad x^4-10x^2y^2+9y^4=280. \quad (2)$

From (1), $\quad x=\frac{11}{2};$

Substituting in (2), $\quad \frac{14641}{16}-\frac{1210y^2}{4}+9y^4=280;$

whence, $\quad y=\frac{3}{2}.$

Therefore the terms are 1, 4, 7, 10.

13. Let $\quad x-3y$, $x-y$, $x+y$, $x+3y$, be the terms required.

Then, $\quad 2x^2+18y^2=200; \quad (1)$

and $\quad 2x^2+2y^2=136. \quad (2)$

Subtracting (2) from (1), $\quad 16y^2=64;$

whence, $\quad y=2;$

Substituting in (2), $\quad 2x^2+8=136;$

whence, $\quad x=8.$

Therefore the terms are 2, 6, 10, 14.

14. Let $\quad x-3y$, $x-y$, $x+y$, $x+3y$ be the numbers required.

Then, $\quad x^2-9y^2=45; \quad (1)$

and $\quad x^2-y^2=77. \quad (2)$

Subtracting (1) from (2), $\quad 8y^2=32;$

whence, $\quad y=2;$

Substituting in (2), $\quad x^2-4=77;$

whence, $\quad x=9.$

Therefore the required terms are 3, 7, 11, 15.

15. Let $\quad x-3y$, $x-y$, $x+y$ and $x+3y$ be the numbers required.

Then, $\quad 2x=17; \quad (1)$

and $\quad 4xy=51. \quad (2)$

From (1), $\quad x=\frac{17}{2};$

Substituting in (2), $\quad 34y=51;$

whence, $\quad y=\frac{3}{2}.$

Therefore the terms are 4, 7, 10, 13.

16. Let $x-3y,\ x-y,\ x+y,\ x+3y$, be the required numbers.

Then, $2x^2+2y^2=164;$ (1)

and $2x^2+18y^2=180.$ (2)

Subtracting (1) from (2), $16y^2=16;$

whence, $y=1;$

Substituting in (1), $2x^2+2=164;$

whence, $x=9.$

Therefore the numbers are 6, 8, 10, 12.

17. Let $x-2y,\ x-y,\ x,\ x+y,\ x+2y$, be the numbers.

Then, $5x=40;$ (1)

and $5x^2+10y^2=410.$ (2)

From (1), $x=8;$

Substituting in (2), $320+10y^2=410;$

whence, $y=3.$

Therefore the numbers are 2, 5, 8, 11, 14.

18. Let $x-3y, x-2y, x-y, x, x+y, x+2y, x+3y$, be the numbers.

Then, $2x-2y=16;$ (1)

and $x^2+3xy=160.$ (2)

From (1), $y=x-8;$ (3)

Substituting in (2), $x^2+3x^2-24x=160;$ (4)

Reducing, $x^2-6x=40;$

whence, $x=10;$

and $y=2.$

Therefore the numbers are 4, 6, 8, 10, 12, 14, 16.

19. $S=\frac{1}{2}n\{2a+(n-1)d\}=n^2.$

Let $n=1$; then, $a=1$;

Let $n=2$; then, $2a+d=4$; $d=4-2=2.$

This example may be solved thus:

Let $n=1$; then, $S=1$, which is the first term.

Let $n=2$; then, $S=4$, which is the sum of the first and second terms; hence, $4-1$, or $3=$ second term, etc. A similar solution will apply to the 20th.

20. $S=\frac{1}{2}n\{2a+(n-1)d\}=\frac{1}{2}n(n+1).$

Let $n=1$; then, $a=1$.

Let $n=2$; then, $2a+d=3$; $d=1$.

Hence the series is 1, 2, 3, 4, etc.

Art. 369. (page 263.)

4. Given, $l = ar^{n-1}$; substituting, $l = 729 \times (\frac{1}{3})^{11} = \frac{1}{243}$.

5. Given, $l = ar^{n-1}$; substituting, $l = 1 \times 2^{n-1} = 2^{n-1}$.

6. Given, $l = ar^{n-1}$; substituting, $l = 2a \times (2a)^{n-1} = (2a)^n$.

7. Given, $l = ar^{n-1}$; substituting, $l = 2 \times (2a)^{n-1} = 2^n a^{n-1}$.

8. Given, $l = ar^{n-1}$; substituting, $l = 4000 \times 2^5 = 128000$.

Art. 370. (page 264.)

7. Given, $S = \dfrac{ar^n - a}{r-1}$. Substituting, $S = \dfrac{1 \times 2^n - 1}{2-1} = 2^n - 1$.

8. Given, $S = \dfrac{ar^n - a}{r-1}$. Substituting, $S = \dfrac{1 \times 3^n - 1}{3-1} = \dfrac{3^n - 1}{2}$.

9. Given, $S = \dfrac{ar^n - a}{r-1}$. Substituting, $S = \dfrac{a \times 2^n - a}{2-1} = (2^n - 1)a$.

10. Given, $S = \dfrac{ar^n - a}{r-1}$. Substituting, $S = \dfrac{1 \times (\frac{1}{2})^n - 1}{\frac{1}{2} - 1} = \dfrac{\frac{1-2^n}{2^n}}{-\frac{1}{2}} = \dfrac{2^n - 1}{2^{n-1}}$.

11. Given, $S = \dfrac{ar^n - a}{r-1}$. Substituting, $S = \dfrac{1 \times (\frac{1}{3})^n - 1}{\frac{1}{3} - 1} = \dfrac{\frac{1-3^n}{3^n}}{-\frac{2}{3}}$ $= \frac{1}{2}\left(\dfrac{3^n - 1}{3^{n-1}}\right)$.

12. Given, $S = \dfrac{ar^n - a}{r-1}$. Substituting, $S = \dfrac{1 \times (-\frac{1}{2})^n - 1}{-\frac{1}{2} - 1} = \dfrac{\frac{\pm 1 - 2^n}{2^n}}{-\frac{3}{2}}$ $= \frac{1}{3}\left(\dfrac{2^n - 1}{2^{n-1}}\right)$, or $\frac{1}{3}\left(\dfrac{2^n + 1}{2^{n-1}}\right)$.

13. Given, $S = \dfrac{ar^n - a}{r-1}$. Substituting, $S = \dfrac{1 \times (2)^{12} - 1}{2-1} = 4095$.

14. Given, $S = \dfrac{ar^n - a}{r-1}$. Substituting, $S = \dfrac{160 \times (\frac{3}{2})^9 - 160}{\frac{3}{2} - 1} = 11981.87\frac{1}{2}$.

Art. 372. (page 265.)

7. Given, $S=\dfrac{a}{1-r}$; $r=\frac{1}{10}$. Substituting, $S=\dfrac{\frac{3}{10}}{1-\frac{1}{10}}=\frac{1}{3}$.

8. Given, $S=\dfrac{a}{1-r}$. The circulate $=\frac{2}{10}+\frac{27}{1000}+\frac{27}{100000}$, etc. The first term is $\frac{27}{1000}$; the ratio, $\frac{1}{100}$. Substituting, $S=\dfrac{\frac{27}{1000}}{1-\frac{1}{100}}=\frac{27}{990}=\frac{3}{110}$; $\frac{3}{110}+\frac{2}{10}=\frac{25}{110}=\frac{5}{22}$.

9. Given, $S=\dfrac{a}{1-r}$; $r=\dfrac{1}{a}$. Substituting, $S=\dfrac{\frac{1}{a}}{1-\frac{1}{a}}=\dfrac{1}{a-1}$.

10. Given, $S=\dfrac{a}{1-r}$; $r=x^{-2}$. Substituting, $S=\dfrac{1}{1-\frac{1}{x^2}}=\dfrac{x^2}{x^2-1}$.

11. Given, $S=\dfrac{a}{1-r}$; $r=-\dfrac{b}{a}$. Substituting, $S=\dfrac{a}{1+\frac{b}{a}}=\dfrac{a^2}{a+b}$.

12. Given, $S=\dfrac{a}{1-r}$; $r=\frac{1}{2}$. Substituting, $S=\dfrac{12}{1-\frac{1}{2}}=24$.

13. Given, $S=\dfrac{a}{1-r}$; $r=\frac{1}{2}$. It bounds 6 feet and falls 6, so that the first term is 12 feet, the second 6, etc.

Then, $S=\dfrac{12}{1-\frac{1}{2}}=24$. Adding the 12 feet it fell before bounding, we have $12+24=36$.

Or we may take the first term, $12+6=18$; the second, $6+3$, or 9, etc.

14. Given, $S=\dfrac{a}{1-r}$; $r=\frac{1}{10}$. Substituting, $S=\dfrac{20}{1-\frac{1}{10}}=22\frac{2}{9}$.

Art. 375. (page 266.)

1. $l=ar^{n-1}$.

Hence, $a=\dfrac{l}{r^{n-1}}$.

2. $l=ar^{n-1}$.

Hence, $r=\sqrt[n-1]{\dfrac{l}{a}}$

3 $a=\dfrac{l}{r^{n-1}}$.

Substituting, $a=\dfrac{512}{2^9}=1$.

4. $r=\sqrt[n-1]{\dfrac{l}{a}}$.

Substituting, $r=\sqrt[7]{\dfrac{109.35}{6.40}}=1\frac{1}{2}$.

Art. 376. (page 267.)

1. Given, $S=\frac{rl-a}{r-1}$. Hence, $rl-a=(r-1)S$; and $a=rl-(r-1)S$.

2. Given, $S=\frac{rl-a}{r-1}$. Then, $rl-a=(r-1)S$; and $l=\frac{(r-1)S+a}{r}$.

3. Given, $S=\frac{rl-a}{r-1}$. Then, $rl-a=rS-S$; and $r=\frac{S-a}{S-l}$.

4. Given, $a=\frac{(r-1)S}{r^n-1}$. Substituting, $a=\frac{(4095)(2-1)}{4096-1}=1$.

5. Given, $r=\sqrt[n-1]{\frac{l}{a}}$. Substituting, $r=\sqrt[9]{\frac{512}{1}}=2$.

Art. 377. (page 267.)

1. Given, $S=\frac{rl-a}{r-1}$; and $a=\frac{l}{r^{n-1}}$. Substituting, $S=\frac{rl-\frac{l}{r^{n-1}}}{r-1}$ $=\frac{lr^n-l}{r^n-r^{n-1}}$.

2. Given, $S=\frac{lr^n-l}{r^n-r^{n-1}}$. Then, $l(r^n-1)=Sr^n-Sr^{n-1}$; and $l=\frac{Sr^{n-1}(r-1)}{r^n-1}$.

3. Given, $S=\frac{ar^n-a}{r-1}$. Then, $a=\frac{S(r-1)}{r^n-1}$. Substituting, $a=\frac{295.24(3-1)}{3^{10}-1}=\frac{59048}{59048}=1$.

Art. 383. (page 270.)

3. Let $n=1$; then, $6^{n-1}=6^0=1$, first term.
Let $n=2$; then, $6^{n-1}=6^1=6$, second term.
Hence the *rate* is 6, and the series, 1, 6, 36, 216, etc.

4. Let $n=1$; then, $3^n=3^1=3$, first term.
Let $n=2$; then, $3^n=3^2=9$, second term.
Hence the *rate* is 3, and the series is 3, 9, 27, etc.

5. Let $n=1$; then, $(2a)^n=2a$, first term.
Let $n=2$; then, $(2a)^n=(2a)^2=4a^2$, second term.
Hence the *rate* is $4a^2\div 2a=2a$, and the series, $2a$, $4a^2$, $8a^3$, etc.

6. $$S=\tfrac{1}{2}(3^n-1).$$

Let $n=1$; then, $\frac{1}{2}(3^n-1)=1$, first term.

Let $n=2$; then, $\frac{1}{2}(3^n-1)=4$; $4-1=3$, second term.

Hence the *rate* is 3, and the series is 1, 3, 9, 27, etc.

7. $$S=a(2^n-1).$$

Let $n=1$; then, $a(2^n-1)=a$, first term.

Let $n=2$; then, $a(2^n-1)=3a$; $3a-a=2a$, second term.

Hence the *rate* is 2, and the series is a, $2a$, $4a$, $8a$, etc.

8. $$S=\frac{2^n-1}{2^{n-1}}.$$

Let $n=1$; then, $\dfrac{2^n-1}{2^{n-1}}=1$, first term.

Let $n=2$; then, $\dfrac{2^n-1}{2^{n-1}}=\frac{3}{2}$; $\frac{3}{2}-1=\frac{1}{2}$, second term.

Let $n=3$; then, $\dfrac{2^n-1}{2^{n-1}}=\frac{7}{4}$; $\frac{7}{4}-\frac{3}{2}=\frac{1}{4}$, third term.

Hence the *rate* is $\frac{1}{2}$, and the series is $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$, etc.

9. Let x^2, xy, y^2 be the numbers.

Then, $$x^2+xy+y^2=28; \quad (1)$$
and $$x^4+x^2y^2+y^4=336. \quad (2)$$

Dividing (2) by (1), $$x^2-xy+y^2=12; \quad (3)$$
Subtracting (3) from (1), $$xy=8; \quad (4)$$
Adding (1) and (4), $$(x+y)^2=36;$$
whence, $$x+y=6; \quad (5)$$
By substitution, $$x=2 \text{ and } y=4.$$

Therefore, 4, 8, and 16 are the numbers.

10. Let x^2, xy, and y^2 be the numbers.

Then, $$x^3y^3=216; \quad (1)$$
and $$x^4+x^2y^2+y^4=364. \quad (2)$$

Extracting the cube root of (1), $$xy=6;$$
whence, $$x=\frac{6}{y};$$
Substituting in (2), $$\frac{1296}{y^4}+36+y^4=364;$$
Reducing, $$y^8-328y^4=-1296;$$
whence, $$y^2=18;$$
and $$x^2=2.$$

Therefore, the numbers are 2, 6, 18.

11. Let x, $\sqrt{xy}$, y be the numbers.

Then, $x+y=10$; (1)

and $x^3+y^3=520$. (2)

Dividing (2) by (1), $x^2-xy+y^2=52$; (3)

Squaring (1), $x^2+2xy+y^2=100$. (4)

Subtracting (3) from (4), $3xy=48$;

whence, $xy=16$.

Therefore, the numbers are 2, 4, and 8.

12. Let x, $\sqrt{xy}$, y be the numbers.

Then, $x+\sqrt{xy}=32$; (1)

and $\sqrt{xy}+y=96$. (2)

Adding (1) and (2), $x+2\sqrt{xy}+y=128$; (3)

Evolving, $\sqrt{x}+\sqrt{y}=8\sqrt{2}$; (4)

Subtracting (1) from (2), $y-x=64$; (5)

Dividing (5) by (4), $\sqrt{y}-\sqrt{x}=4\sqrt{2}$; (6)

Subtracting (6) from (4), $\sqrt{x}=2\sqrt{2}$;

whence, $x=8$;

and $y=72$;

and $\sqrt{xy}=24$.

13. Let x^2, xy, and y^2 be the numbers.

Then, $x^3y^3=216$; (1)

and $x^4+y^4=153$. (2)

From (1), $2x^2y^2=72$; (3)

Adding (3) and (2), $x^4+2x^2y^2+y^4=225$;

whence, $x^2+y^2=15$;

Subtracting (3) from (2), $x^4-2x^2y^2+y^4=81$;

whence, $y^2-x^2=9$;

and $x^2=3$ and $y^2=12$.

14. Let x^2, xy, and y^2 be the numbers.

Then, $x^2+xy+y^2=39$; (1)

and $x^3y+xy^3=270$. (2)

Multiplying (1) by xy, $x^3y+x^2y^2+xy^3=39xy$; (3)

Subtracting (2) from (3), $x^2y^2=39xy-270$; (4)

Completing the square, $x^2y^2-39xy+\frac{1521}{4}=\frac{441}{4}$;

whence, $xy=9$; (5)

and $x=\dfrac{9}{y}$;

Substituting in (1), $\dfrac{81}{y^2}+9+y^2=39$; (6)

whence, $y^2=27$;

and $x^2=3$.

15. Let x^2, xy, and y^2 be the numbers.

Then,	$x^2+xy+y^2=52;$	(1)
and	$x^4+x^2y^2+y^4=1456.$	(2)
Dividing (2) by (1),	$x^2-xy+y^2=28;$	(3)
Subtracting (3) from (1),	$xy=12;$	(4)
Adding (4) to (1),	$x^2+2xy+y=64;$	(5)
Evolving,	$x+y=8;$	
whence,	$x=2$, and $y=6$.	

Therefore the numbers are 4, 12, 36.

16. Let x, $\sqrt{xy}$, y be the numbers.

Then,	$x+y=30;$	(1)
and	$xy=144.$	(2)
From (1),	$x=30-y;$	
Substituting in (2),	$30y--y^2=144;$	
whence,	$y=24;$	
and	$x=6;$	
and	$\sqrt{xy}=12.$	

17. Let $\frac{x^2}{y}$, x, y, $\frac{y^2}{x}$ be the numbers.

Then,	$\frac{x^2}{y}+y=20;$	(1)
and	$x+\frac{y^2}{x}=60;$	(2)
Clearing (1) of fractions,	$x^2+y^2=20y;$	(3)
Clearing (2) of fractions.	$x^2+y^2=60x;$	(4)
Equating,	$60x=20y;$	
or,	$3x=y;$	
Substituting in (1),	$\frac{x^2}{3x}+3x=20;$	
Reducing,	$x=6;$	
and	$y=18.$	

Therefore the numbers are 2, 6, 18, 54.

18. Let x, xy, xy^2, xy^3 be the numbers.

Then,	$x+xy=10;$	(1)
and	$xy^2+xy^3=160.$	(2)
Dividing (2) by (1),	$y^2=16;$	
whence,	$y=4;$	
and	$x=2.$	

Therefore the numbers are 2, 8, 32, 128.

MISCELLANEOUS EXAMPLES.

13. $15x-\{4-[3-5x-(3x-7)]\}=15x-4+[3-5x-(3x-7)]$ $=15x-4+3-5x-(3x-7)=10x-1-3x+7=7x+6.$

14. $2x-[3y-\{4x-(5y-6x+7y)\}]=2x-3y+\{4x-(12y-6x)\}$ $=2x-3y+4x-(12y-6x)=6x-3y-12y+6x=12x-15y.$

15. $a-[5b-\{a-(5c-\overline{2c-b}-4b)+2a-(a-\overline{2b+c})\}]=a-[5b$ $-\{3a-(3c-3b)-(a-2b-c)\}]=a-[5b-\{2a-2c+5b\}]=a-[2c$ $-2a]=3a-2c.$

16.
$$
\begin{aligned}
&(a-b)^3=(a-b)(a^2-2ab+b^2)=(b-a)(2ab-a^2-b^2)\\
&b^3-a^3= \qquad\qquad\qquad\qquad\qquad\quad\; (b-a)(ab+a^2+b^2)\\
&\text{Adding,} \qquad\qquad\qquad\qquad\qquad\quad\;\; 3ab(b-a)
\end{aligned}
$$

17.
$$
\begin{aligned}
&(a^2+ab+b^2)^2=a^4+2a^3b+3a^2b^2+2ab^3+b^4\\
&(a^2-ab+b^2)^2=a^4-2a^3b+3a^2b^2-2ab^3+b^4\\
&\text{Subtracting,} \qquad 4a^3b+4ab^3=4ab(a^2+b^2)
\end{aligned}
$$

18. $(a+b+c)^3-(a^3+b^3+c^3)=a^3+b^3+c^3+3a^2b+3ab^2+3a^2c$ $+3ac^2+3b^2c+3bc^2+6abc-a^3-b^3-c^3=3a^2b+3ab^2+3a^2c+3ac^2+3b^2c$ $+3bc^2+6abc.$

$$3a^2b+3ab^2=3ab(a+b);\ 3a^2c+3abc=3ac(a+b);$$
$$3abc+3b^2c=3bc(a+b);\ 3ac^2+3bc^2=3c^2(a+b).$$
$$ab+ac+bc+c^2=(b+c)(a+c).$$
$$\therefore\ 3ab(a+b)+3ac(a+b)+3bc(a+b)+3c^2(a+b)=3(a+b)(b+c)(a+c).$$

19. $(a^n-2)(a^n+2)(a^n+3)(a^n-3)=?$
$$(a^n-2)(a^n+2)=a^{2n}-4;\ (a^n+3)(a^n-3)=a^{2n}-9.$$
$$(a^{2n}-4)(a^{2n}-9)=a^{4n}-13a^{2n}+36.$$

20. a^4+b^4 has no factors.
$$a^5-b^5=(a-b)(a^4+a^3b+a^2b^2+ab^3+b^4).$$
$$a^5+b^5=(a+b)(a^4-a^3b+a^2b^2-ab^3+b^4).$$
$$a^{4n}-b^{4n}=(a^{2n}+b^{2n})(a^n-b^n)(a^n+b^n).$$

21. $a^6-b^6=(a+b)(a^2-ab+b^2)(a^2+ab+b^2)(a-b).$

a^6+b^6 has no factors.
$$a^8-b^8=(a^4+b^4)(a^2+b^2)(a+b)(a-b).$$
a^8+b^8 has no factors.
$$(a^{8n}-b^{8n})=(a^{4n}+b^{4n})(a^{2n}+b^{2n})(a^n+b^n)(a^n-b^n).$$

22. $$a^2+9ab+20b^2=(a+4b)(a+5b).$$
$$a^nc^n-b^nc^n+a^nd^n-b^nd^n=(a^n-b^n)(c^n+d^n).$$

23. $a^2+8a+15=(a+3)(a+5)$.
$a^2+9a+20=(a+4)(a+5)$.
The only common factor is $a+5$.

24. $5(x^2-x+1)=5(x^2-x+1)$.
$4(x^6-1)=4(x^2-x+1)(x^2+x+1)(x-1)(x+1)$.
$2(x^3+1)=2(x^2-x+1)(x+1)$.
$\therefore$ x^2-x+1 is the greatest common divisor.

25. $$x^3+1=(x^2-x+1)(x+1).$$
$$(x^3-1)=(x^2+x+1)(x-1).$$
$$x^2-x+1=x^2-x+1.$$
$(x^3+1)(x^3-1)=x^6-1$, the least common multiple.

26. $a^2+1=a^2+1$.
$a^2-1=(a+1)(a-1)$.
$a^4+1=a^4+1$.
$a^8-1=(a^4+1)(a^2+1)(a+1)(a-1)$.
a^8-1 contains all the others, therefore is least common multiple.

27. $$\frac{x^2+3x+2}{x^2+6x+5}=\frac{(x+1)(x+2)}{(x+1)(x+5)}=\frac{x+2}{x+5}.$$

$$\frac{x^2+10x+21}{x^2-2x-15}=\frac{(x+3)(x+7)}{(x+3)(x-5)}=\frac{x+7}{x-5}.$$

28. $$\frac{x^2+(a+b)x+ab}{x^2+(a+c)x+ac}=\frac{(x+a)(x+b)}{(x+a)(x+c)}=\frac{x+b}{x+c}.$$

$$\frac{x^4+a^2x^2+a^4}{x^6-a^6}=\frac{x^4+a^2x^2+a^4}{(x^2-a^2)(x^4+a^2x^2+a^4)}=\frac{1}{x^2-a^2}.$$

29. $$\frac{a-b}{b}+\frac{2a}{a-b}-\frac{a^3+a^2b}{a^2b-b^3}=\frac{a^3-a^2b-ab^2+b^3}{a^2b-b^3}+\frac{2a^2b+2ab^2}{a^2b-b^3}$$

$$-\frac{a^3+a^2b}{a^2b-b^3}=\frac{ab^2+b^3}{a^2b-b^3}=\frac{b}{a-b}.$$

30. $$\frac{a^2}{(a-b)(a-c)}+\frac{b^2}{(b-a)(b-c)}+\frac{c^2}{(c-a)(c-b)}=\frac{a^2(b-c)}{(a-b)(a-c)(b-c)}$$

$$-\frac{b^2(a-c)}{(a-b)(b-c)(a-c)}+\frac{c^2(a-b)}{(a-b)(b-c)(a-c)}$$

$$=\frac{a^2b-a^2c-ab^2+b^2c+ac^2-bc^2}{a^2b-ab^2+b^2c-bc^2-a^2c+ac^2}=1.$$

NOTE.—In the second fraction the sign of one of the factors of the denominator is changed, which changes the sign of the fraction. In the third fraction the signs of both factors of the denominator are changed, therefore the sign of the fraction remains the same.

31. $$\left(b^n+\frac{a^{2n}}{b^n}\right)\left(a^n-\frac{b^{2n}}{a^n}\right)=\left(\frac{b^{2n}+a^{2n}}{b^n}\right)\left(\frac{a^{2n}-b^{2n}}{a^n}\right)=\frac{a^{4n}-b^{4n}}{a^nb^n}.$$

32. $$\frac{x^2+xy}{x^2+y^2}\times\left(\frac{x}{x-y}-\frac{y}{x+y}\right)=\left(\frac{x^2+xy}{x^2+y^2}\right)\left(\frac{x^2+xy-xy+y^2}{x^2-y^2}\right)=\frac{x^2+xy}{x^2+y^2}$$

$$\times\frac{x^2+y^2}{x^2-y^2}=\frac{x}{x-y}.$$

33.
$$\begin{array}{llllll}
\frac{x}{a}-\frac{a}{x}+\frac{y}{b}-\frac{b}{y} & & & & & \\
\frac{x}{a}-\frac{a}{x}-\frac{y}{b}+\frac{b}{y} & & & & & \\
\hline
\frac{x^2}{a^2}-1+\frac{xy}{ab}-\frac{bx}{ay} & & & & & \\
\quad -1 & +\frac{a^2}{x^2}-\frac{ay}{bx}+\frac{ab}{xy} & & & & \\
\quad -\frac{xy}{ab} & +\frac{ay}{bx} & -\frac{y^2}{b^2}+1 & & & \\
\quad +\frac{bx}{ay} & -\frac{ab}{xy} & +1-\frac{b^2}{y^2}. & & & \\
\hline
\frac{x^2}{a^2}+\frac{a^2}{x^2}-\frac{y^2}{b^2}-\frac{b^2}{y^2} & & & & &
\end{array}$$

34. $\dfrac{x^2+(a+c)x+ac}{x^2+(b+c)x+bc} \div \dfrac{x^2-a^2}{x^2-b^2} = \dfrac{(x+a)(x+c)}{(x+b)(x+c)} \div \dfrac{(x+a)(x-a)}{(x+b)(x-b)}$

$= \dfrac{x-b}{x-a}.$

35. $\left(1+\dfrac{x}{y}\right)\left(1-\dfrac{x}{y}\right) \div \dfrac{y}{x^2+y^2} = \left(\dfrac{y+x}{y}\right)\left(\dfrac{y-x}{y}\right) \times \dfrac{x^2+y^2}{y} = \dfrac{y^4-x^4}{y^3}.$

36. $\left(a^3-\dfrac{1}{a^3}\right) \div \left(a-\dfrac{1}{a}\right) = a^2+1+\dfrac{1}{a^2} = \dfrac{a^4+a^2+1}{a^2}.$

37. $\left(\dfrac{a^2}{x^2}+1+\dfrac{x^2}{a^2}\right) \div \left(\dfrac{a}{x}-1+\dfrac{x}{a}\right) = \dfrac{a}{x}+1+\dfrac{x}{a} = \dfrac{a^2+ax+x^2}{ax}.$

38. $\dfrac{x-1+\dfrac{6}{x-6}}{x-2+\dfrac{3}{x-6}} = \dfrac{\dfrac{x^2-7x+12}{x-6}}{\dfrac{x^2-8x+15}{x-6}} = \dfrac{x^2-7x+12}{x^2-8x+15} = \dfrac{(x-3)(x-4)}{(x-3)(x-5)} = \dfrac{x-4}{x-5}.$

39. $1+\dfrac{x}{1+x+\dfrac{2x^2}{1-x}} = 1+\dfrac{x}{\dfrac{1+x^2}{1-x}} = 1+\dfrac{x-x^2}{1+x^2} = \dfrac{1+x}{1+x^2}.$

40. $\dfrac{1}{1+\dfrac{a}{1+a+\dfrac{2a^2}{1-a}}} = \dfrac{1}{1+\dfrac{a}{\dfrac{1+a^2}{1-a}}} = \dfrac{1}{1+\dfrac{a-a^2}{1+a^2}} = \dfrac{1}{\dfrac{1+a}{1+a^2}} = \dfrac{1+a^2}{1+a}.$

41. $\dfrac{3x-9}{x^2-7x+12}$, when $x=3$. Reducing, $\dfrac{3}{x-4}$; substituting, $\dfrac{3}{-1}$ $= -3.$

42. $\dfrac{a^2-x^2}{(a-x)^2}$, when $x=a$. Reducing, $\dfrac{a+x}{a-x}$; substituting, $\dfrac{2a}{0} = \infty.$

43. $\dfrac{ax^2+ac^2-2acx}{bx^2-2bcx+bc^2}$, when $x=c$. Dividing by $(x-c)^2$, we have $\dfrac{a}{b}.$

NOTE.—This value is the same, whatever is the value of x.

49. $\frac{3+x}{3-x}-\frac{2+x}{2-x}-\frac{1+x}{1-x}=1.$ Clearing of fractions, $6-7x+x^3-6$ $+5x+2x^2-x^3-6-x+4x^2-x^3=6-11x+6x^2-x^3$; reducing, $8x=12$; $x=\frac{3}{2}$.

50. $\frac{x^2-x+1}{x-1}+\frac{x^2+x+1}{x+1}=2x.$ $x^3+1+x^3-1=2x^3-2x$; $x=0$.

53.
$$\frac{1}{x-a}-\frac{1}{x-b}=\frac{a-b}{x^2-ab}.$$

Reducing first member, $\frac{a-b}{x^2-(a+b)x+ab}=\frac{a-b}{x^2-ab}$;

Clearing of fractions, $x^2-ab=x^2-(a+b)x+ab$;

$$x=\frac{2ab}{a+b}.$$

56.
$$3x+9y=2.4; \quad (1)$$
$$.21x-.06y=.03. \quad (2)$$

Dividing (1) by 3, $x+3y=.8$; (3)

Multiplying (3) by .02, $.02x+.06y=.016$; (4)

Adding (4) and (2), $.23x=.046$;

$x=.2$;

$y=.2$.

57.
$$.3x+.125y=x-6; \quad (1)$$
$$3x-.5y=28-.25y. \quad (2)$$

Reducing (1), $-.7x+.125y=-6$; (3)

Reducing (2), $3x-.25y=28$; (4)

Multiplying (3) by 2, $1.4x+.25y=-12$. (5)

Adding (4) and (5), $1.6x=16$;

whence, $x=10$;

and $y=8$.

58.
$$x+y=a+b; \quad (1)$$
$$bx+ay=2ab. \quad (2)$$

Multiplying (1) by b, $bx+by=ab+b^2$; (3)

Subtracting (3) from (2), $(a-b)y=ab-b^2$;

whence, $y=b$;

and $x=a$.

59.

$$\frac{x}{a}+\frac{y}{b}=c; \qquad (1)$$

$$\frac{x}{b}-\frac{y}{a}=0. \qquad (2)$$

Multiplying (2) by $\frac{b}{a}$, $\quad \frac{x}{a}-\frac{by}{a^2}=0; \qquad (3)$

Subtracting (3) from (1), $\quad \left(\frac{1}{b}+\frac{b}{a^2}\right)y=c;$

whence, $\quad y=\frac{a^2bc}{a^2+b^2};$

and $\quad x=\frac{ab^2c}{a^2+b^2}.$

60.

$$a(x+y)+b(x-y)=1; \qquad (1)$$

$$a(x-y)+b(x+y)=1. \qquad (2)$$

Expanding (1), $\quad ax+ay+bx-by=1; \qquad (3)$

Expanding (2), $\quad ax-ay+bx+by=1. \qquad (4)$

Adding (3) and (4), $\quad 2(a+b)x=2;$

whence, $\quad x=\frac{1}{a+b};$

Substituting in (4), $\quad by-ay=0;$

whence, $\quad y=0.$

61.

$$4x-5y+z=6; \qquad (1)$$

$$7x-11y+2z=9; \qquad (2)$$

$$x+y+3z=12. \qquad (3)$$

Multiplying (1) by 2, $\quad 8x-10y+2z=12; \qquad (4)$

Subtracting (2) from (4), $\quad x+y=3; \qquad (5)$

Substituting (5) in (3), $\quad 3z=9;$

whence, $\quad z=3;$

Substituting in (1), $\quad 4x-5y=3; \qquad (6)$

Multiplying (5) by 5, $\quad 5x+5y=15. \qquad (7)$

Adding (6) and (7), $\quad 9x=18;$

whence, $\quad x=2;$

and $\quad y=1.$

62.
$$y+z=a; \quad (1)$$
$$x+z=b; \quad (2)$$
$$x+y=c. \quad (3)$$

Subtracting (1) from (2), $x-y=b-a;$ (4)

Adding (3) and (4), $x=\dfrac{b+c-a}{2};$

Subtracting (4) from (3), $y=\dfrac{a+c-b}{2};$

Substituting in (1), $z=\dfrac{a+b-c}{2}.$

63.
$$y+z-x=a; \quad (1)$$
$$x+z-y=b; \quad (2)$$
$$x+y-z=c. \quad (3)$$

Adding (1) and (2), $z=\dfrac{a+b}{2};$

Adding (2) and (3), $x=\dfrac{b+c}{2};$

Adding (1) and (3), $y=\dfrac{a+c}{2}.$

64.
$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1; \quad (1)$$
$$\frac{x}{a}+\frac{y}{c}+\frac{z}{b}=1; \quad (2)$$
$$\frac{x}{b}+\frac{y}{a}+\frac{z}{c}=1. \quad (3)$$

Subtracting (1) from (2), $\dfrac{y}{b}+\dfrac{z}{c}=\dfrac{y}{c}+\dfrac{z}{b};$ (4)

whence, $\dfrac{(b-c)y}{bc}=\dfrac{(b-c)z}{bc};$

and $y=z;$

Subtracting (1) from (3), $\dfrac{x}{a}+\dfrac{y}{b}=\dfrac{x}{b}+\dfrac{y}{a};$ (5)

whence, $\left(\dfrac{b-a}{ab}\right)x=\left(\dfrac{b-a}{ab}\right)y;$

and $x=y;$

Substituting in (1), $\dfrac{y}{a}+\dfrac{y}{b}+\dfrac{y}{c}=1;$

whence, $x, y, \text{ or } z=\dfrac{abc}{ab+bc+ac}.$

65.

$$\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=3; \qquad (1)$$

$$\frac{a}{x}+\frac{b}{y}-\frac{c}{z}=1; \qquad (2)$$

$$\frac{2a}{x}-\frac{b}{y}-\frac{c}{z}=0. \qquad (3)$$

From (3), $\frac{2a}{x}=\frac{b}{y}+\frac{c}{z}$;

Substituting in (1), $\frac{3a}{x}=3$;

whence, $x=a$;

Subtracting (2) from (1), $\frac{2c}{z}=2.$

whence, $z=c$;

Substituting in (1), $\frac{b}{y}=1$;

whence, $y=b.$

66. $\sqrt{\frac{a^{m+n}}{a^{m-n}}}=\sqrt{a^{2n}}=a^n$; $\left(\frac{a}{2}\sqrt[4]{\frac{a}{2}}\right)^{\frac{1}{5}}=\left(\sqrt[4]{\frac{a^5}{2^5}}\right)^{\frac{1}{5}}=\sqrt[4]{\frac{a}{2}}=\frac{1}{2}\sqrt[4]{8a}.$

73. $\left\{-(x^3)^{\frac{1}{2}}\right\}^{-\frac{1}{3}}\times\left\{-(-x)^{-3}\right\}^{\frac{1}{2}}=\left(-x^{\frac{3}{2}}\right)^{-\frac{1}{3}}\times(x^{-3})^{\frac{1}{2}}=\left(-x^{-\frac{1}{2}}\right)\cdot$
$\left(x^{-\frac{3}{2}}\right)=-x^{-2}.$

74. $x^2+xy+y^2=\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)^2+\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)+\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)^2$

$$=\frac{4+2\sqrt{3}}{4-2\sqrt{3}}+1+\frac{4-2\sqrt{3}}{4+2\sqrt{3}}.$$

Reducing to a com. denominator, $\frac{16+16\sqrt{3}+12}{16-12}+1+\frac{16-16\sqrt{3}+12}{16-12}$

$$\frac{56}{4}+1=15.$$

75. Given, $\sqrt{(x-a)}=\frac{a}{\sqrt{(x+a)}}$. Clearing of fractions, $\sqrt{(x^2-a^2)}$ $=a$; squaring, $x^2-a^2=a^2$; then, $x^2=2a^2$; and $x=\pm a\sqrt{2}.$

76. Given, $\frac{x-4}{2x+1}=\frac{2x-1}{x+4}$. Clearing of fractions, $x^2-16=4x^2-1$; whence, $3x^2=-15$; and $x=\pm\sqrt{-5}$.

77. Given, $\frac{\sqrt{x}-2}{\sqrt{x}+10}=\frac{\sqrt{x}-1}{\sqrt{x}+23}$. Clearing of fractions, $x+21\sqrt{x}-46=x+9\sqrt{x}-10$; whence, $12\sqrt{x}=36$; and $x=9$.

78. Given, $4x-\frac{12-x}{x-3}=22$. Clearing of fractions, $4x^2-12x-12+x=22x-66$; reducing, $x^2-\frac{33}{4}x=-\frac{54}{4}$; whence, $x=6$, or $2\frac{1}{4}$.

79. Given, $\frac{2x+11}{x}=5-\frac{x-5}{3}$. Clearing of fractions, $6x+33=15x-x^2+5x$; transposing, $x^2-14x=-33$; whence. $x=11$, or 3.

80.
$$\frac{x-2}{x-3}-\frac{x-4}{x-1}=\frac{14}{15};$$

Clearing of fractions, $15(x^2-3x+2-x^2+7x-12)=14(x^2-4x+3)$;

Reducing, $$x^2-\frac{58}{7}x=-\frac{96}{7};$$

whence, $$x=6, \text{ or } 2\tfrac{2}{7}.$$

81.
$$\frac{x-3}{x-2}-\frac{x-1}{x-4}=-\frac{6}{5}.$$

Clearing of fractions, $5(x^2-7x+12-x^2+3x-2)=-6(x^2-6x+8)$;

Reducing, $$x^2-\frac{28}{3}x=-\frac{49}{3};$$

whence, $$x=7, \text{ or } 2\tfrac{1}{3}.$$

82.
$$\frac{x-1}{x-4}-\frac{x-3}{x-2}=\frac{11}{12}.$$

Clearing of fractions, $12(x^2-3x+2-x^2+7x-12)=11(x^2-6x+8)$.

Reducing, $$x^2-\frac{114x}{11}=-\frac{208}{11};$$

whence, $$x=8, \text{ or } 2\tfrac{4}{11}.$$

83.
$$\frac{x+1}{x+2}+\frac{x-1}{x-2}=\frac{2x-1}{x-1}.$$

Clearing of fractions, $x^3-2x^2-x+2+x^3-3x+2=2x^3-x^2-8x+4;$
Reducing, $x^2-4x=0;$
whence, $x=4$, or 0.

84.
$$\frac{x-2}{x+2}+\frac{x+2}{x-2}=\frac{2(x+3)}{x-3}.$$

Clearing of fractions,
$$x^3-7x^2+16x-12+x^3+x^2-8x-12=2x^3-8x+6x^2-24;$$
Reducing, $12x^2-16x=0,$
whence, $x=1\frac{1}{3}$, or 0.

85.
$$\frac{a-\sqrt{(a^2-x^2)}}{a+\sqrt{(a^2-x^2)}}=b.$$

Clearing of fractions, $a-\sqrt{(a^2-x^2)}=ab+b\sqrt{(a^2-x^2)};$
Transposing, $a(1-b)=(1+b)\sqrt{(a^2-x^2)};$
Squaring, $a^2-2a^2b+a^2b^2=a^2+2a^2b+a^2b^2-(1+b)^2x^2;$
Reducing, $(1+b)^2x^2=4a^2b;$
whence, $x=\dfrac{2a\sqrt{b}}{1+b}.$

86. Given, $x+y=4(x-y);$ (1) $xy=15.$ (2)

From (1), $x=\dfrac{5y}{3};$ substituting in (2), $\dfrac{5y^2}{3}=15;$ whence, $y=3;$ and $x=5$.

87. Given, $x^4+y^4=97;$ (1) $4x^2=9y^2.$ (2)

Extracting square root of (2), $2x=3y;$ whence, $x=\dfrac{3y}{2};$ substituting in (1), $\dfrac{81y^4}{16}+y^4=97;$ whence, $97y^4=97\times 16;$ and $y^4=16;$ whence $y=\pm 2;$ and $x=\pm 3$.

88. Given, $x+y=3(x-y);$ (1) $x^3-y^3=56.$ (2)

From (1), $x=2y;$ Substituting in (2) $8y^3-y^3=56;$ whence, $y=2;$ and $x=4$.

89. Given, $\dfrac{x+a}{x-a}-\dfrac{x-a}{x+a}=\dfrac{b+x}{b-x}-\dfrac{b-x}{b+x}.$ Clearing of fractions, $4ab^2x-4ax^3=4bx^3-4a^2bx;$ dividing by $4x$, $ab^2-ax^2=bx^2-a^2b;$ whence, $x^2=ab;$ and $x=\pm\sqrt{ab}.$

90. Given, $x-y=1$; (1) $x^2-xy+y^2=21$. (2)
From (1), $x=y+1$; substituting in (2), $y^2+2y+1-y^2-y+y^2=21$; reducing, $y^2+y=20$; whence, $y=4$, or -5; and $x=5$, or -4.

NOTE.—Or square first, subtract from second, etc.

91.

	$3x+2y=5xy;$	(1)
	$15x-4y=4xy.$	(2)
Multiplying (1) by 2,	$6x+4y=10xy;$	(3)
Adding (2) and (3),	$21x=14xy;$	(4)
Dividing by $7x$,	$y=\frac{3}{2};$	
and	$x=\frac{2}{3}.$	

The zero values may be found by factoring (4), as follows:

	$7x(3-2y)=0;$
Dividing by $3-2y$,	$7x=0;$
whence,	$x=0;$
and	$y=0.$

92.

	$x+y=xy;$	(1)
	$ax=by.$	(2)
From (2),	$x=\frac{by}{a};$	
Substituting in (1),	$\frac{by}{a}+y=\frac{by^2}{a};$	
Clearing of fractions,	$by+ay=by^2;$	
Factoring,	$(a+b-by)y=0;$	
then,	$y=\frac{a+b}{b}$, or $0;$	
and	$x=\frac{a+b}{a}$, or $0.$	

93.

	$\frac{x}{a}+\frac{y}{b}=2;$	(1)
	$x^2+y^2=ax+by.$	(2)
Clearing (1) of fractions,	$bx+ay=2ab;$	(3)
From (3),	$y=\frac{2ab-bx}{a};$	
Substituting in (2),	$(a^2+b^2)x^2-(3ab^2+a^3)x=-2a^2b^2,$	
whence,	$x=a$, or $\frac{2ab^2}{a^2+b^2};$	
and	$y=b$, or $\frac{2a^2b}{a^2+b^2}.$	

94.

$$x^2+xy=28; \quad (1)$$
$$xy-y^2=3. \quad (2)$$

Let $y=vx$;

Substituting in (1), $x^2+vx^2=28$; (3)

Substituting in (2), $vx^2-v^2x^2=3$; (4)

From (3), $x^2=\dfrac{28}{1+v}$; (5)

From (4), $x^2=\dfrac{3}{v-v^2}$; (6)

Equating values of v, $\dfrac{28}{1+v}=\dfrac{3}{v-v^2}$; (7)

Clearing of fractions, $28v-28v^2=3+3v$;

whence, $v=\frac{1}{7}$, or $\frac{3}{4}$;

Substituting in (5), $x^2=\dfrac{28}{1+\frac{1}{7}}$, or $\dfrac{28}{1+\frac{3}{4}}$;

whence, $x=\pm\frac{7}{2}\sqrt{2}$, or ± 4;

and $y=\frac{1}{2}\sqrt{2}$, or ± 3.

95.

$$x+y+\sqrt{(x+y)}=12; \quad (1)$$
$$x^2+y^2=41. \quad (2)$$

Completing the square in (1), $x+y+\sqrt{x+y}+\frac{1}{4}=\frac{49}{4}$; (3)

whence, $\sqrt{x+y}=3$, or -4;

and $x+y=9$, or 16; (4)

From the first value, $x=9-y$;

Substituting in (2), $81-18y+y^2+y^2=41$; (5)

whence, $y=4$, or 5;

and $x=5$, or 4.

96.

$$x^2+y^2+2x+2y=23; \quad (1)$$
$$xy=6. \quad (2)$$

Multiplying (2) by 2, $2xy=12$; (3)

Adding (1) and (3), $(x+y)^2+2(x+y)=35$; (4)

Completing the square, $(x+y)^2+2(x+y)+1=36$;

whence, $x+y=5$;

and $x=3$, or 2;

and $y=2$, or 3.

97.

$$x^2-y^2 : x-y :: 6 : 1; \qquad (1)$$
$$xy=8. \qquad (2)$$

By Theorem I., $x^2-y^2=6(x-y)$;
Dividing by $x-y$, $x+y=6$;
whence, $x=4$;
and $y=2$.

98.

$$x^3-y^3 : (x-y)^3 :: 61 : 1; \qquad (1)$$
$$xy=20. \qquad (2)$$

By Theorem I., $x^3-y^3=61(x-y)^3$; (3)
Dividing (3) by $x-y$, $x^2+xy+y^2=61x^2-122xy+61y^2$; (4)
Substituting (2), $x^2+y^2+20=61x^2+61y^2-2440$; (5)
Reducing, $60x^2+60y^2=2460$;
Dividing by 60, $x^2+y^2=41$; (6)
Adding 2 times (2), $x^2+2xy+y^2=81$; (7)
Extracting the square root, $x+y=9$;
whence, $x=5$, or 4;
and $y=4$, or 5.

99.

$$x^3+y^3 : x^3-y^3 :: 210 : 114; \qquad (1)$$
$$xy^3=24. \qquad (2)$$

By composition and division, $2x^3 : 2y^3 :: 324 : 96$; (3)
By Theorem I., $96 \times 2x^3 = 324 \times 2y^3$;
Reducing, $27y^3=8x^3$;
Extracting the cube root, $3y=2x$;
whence, $x=\dfrac{3y}{2}$;
Substituting in (2), $\dfrac{3y^4}{2}=24$;
whence, $y=2$;
and $x=3$.

100.

$$x^2+y^2=34; \qquad (1)$$
$$x^2-y^2+\sqrt{(x^2-y^2)}=20. \qquad (2)$$

Completing the square in (2), $x^2-y^2+\sqrt{(x^2-y^2)}+\frac{1}{4}=\frac{81}{4}$;
whence, $\sqrt{x^2-y^2}=4$, or -5;
and $x^2-y^2=16$, or 25; (3)
Adding (1) and (3), $x^2=25$;
whence, $x=\pm 5$;
and $y=\pm 3$.

MISCELLANEOUS PROBLEMS.

1. Let x = the time; and $30-x$ = the number of days remaining in month.

Then, $30-x+10=x$; whence, $x=20$.

2. Let x = A's money; and y = B's money.

Then, $x+10=y-10+6$, (1); and $x+y=40$, (2).

Reducing (1), $x-y=-14$, (3); adding (3) and (2), $x=13$; and $y=27$.

3. Let x = the age of youngest; and 4 = common difference; then, $x+5\times 4$ = the age of eldest.

Then, $x+20=3x$; whence, $x=10$. Hence, their ages are 10, 14, 18, 22, 26, 30 years.

4. Let x = the price; then, $\frac{x}{4}$ = what each paid by first condition; and $\frac{x}{6}$ = what each paid by second condition.

Then, $\frac{x}{4}-\frac{x}{6}=1.75$; clearing of fractions, $6x-4x=42$; whence, $x=21$.

5. Let x = the number of pounds in the pudding; then, $\frac{2x}{9}$ = the number of pounds of flour; and $\frac{3x}{9}$ = the number of pounds of raisins; and $\frac{4x}{9}$ = the number of pounds of suet.

Then, $\frac{6x}{9}+\frac{18x}{9}+\frac{32x}{9}=28$; whence, $x=4\frac{1}{2}$; then, $\frac{6x}{9}=3$d.; and $\frac{18x}{9}$ $=9$d.; and $\frac{32x}{9}=1$s. 4d.

6. Let x = the price per dozen; then, $\frac{x}{12}$ = the price of one; and $\frac{8}{\frac{x}{12}}$ $=\frac{96}{x}$ = the number bought for eight pence.

Then, $6x=\frac{96}{x}$; whence, $x=4$.

7. Let x = the tens' digit; and $x+2$ = the units' digit; and $10x+x+2$ = the number.

Then, $10x+20+x : 10x+x+2 :: 7 : 4$; by Theorem I., $44x+80=77x+14$; whence, $x=2$; and $x+2=4$. Hence the number is 24.

8. Let x = the lesser bill; $\frac{4x}{3}$ = the greater bill.

Then, $5-x-\frac{4x}{3}=\frac{x}{6}$; clearing of fractions, $30-6x-8x=x$; whence, $x=2$; and $\frac{4x}{3}=2\frac{2}{3}$.

9. Let x = the value of a share. Then, $\frac{175x}{2}=85x+250$; whence, $x=100$.

10. Let $\frac{x}{y}$ = the fraction. Then, $\frac{4x}{y+3}=\frac{2x}{y}$, (1); and $\frac{x+2}{4y}=\frac{x}{2y}$, (2).

Clearing (1) of fractions, $4xy=2xy+6x$, (3); clearing (2) of fractions, $x+2=2x$, (4); whence, $x=2$; from (3), $2xy=6x$; whence, $y=3$; and $\frac{x}{y}=\frac{2}{3}$.

11. Let x = the number of sheep; $35x+\frac{1.50x}{20}$ = the whole cost.

Then, $40x=35x+\frac{1.50x}{20}+394$; clearing of fractions, $800x=700x+1.50x+7880$; whence, $98.50x=7880$; and $x=80$.

12. Let x = the side of the square; and x^2+31 = the number of men.

Then, $(x+1)^2=x^2+31+24$; expanding and reducing, $2x=54$; and $x=27$; whence, $x^2+31=760$.

13. Let x = the whole number of pounds; then, $\frac{x}{2}+6$ = the number of pounds of saltpetre; $\frac{x}{3}-5$ = the number of pounds of sulphur; $\frac{x}{4}-3$ = the number of pounds of charcoal.

Then, $\frac{x}{2}+6+\frac{x}{3}-5+\frac{x}{4}-3=x$; whence, $\frac{x}{12}=2$; and $x=24$; then, $\frac{x}{2}\times 6=18$; $\frac{x}{3}-5=3$; $\frac{x}{4}-3=3$.

14. Let x = the time required by second. Then, $\frac{1}{20}+\frac{1}{x}=\frac{1}{12}$; clearing of fractions, $12x+240=20x$; whence, $x=30$.

15. Let x = the number to which the parts become equal; then, $x-2$ = the first part; $x+3$ = the second part; $\frac{x}{4}$ = the third part; and $5x$ = the fourth part.

Then, $x-2+x+3+\frac{x}{4}+5x=88$; whence, $\frac{29x}{4}=87$; and $x=12$.

Hence, the parts are 10, 15, 3 and 60.

16. Let x = the money each had at first.

Then, $\frac{x+100}{2}=\frac{1}{2}\left(x-100+\frac{x+100}{2}\right)$;

Clearing of fractions, $2x+200=2x-200+x+100$;

whence, $x=300$.

17. Let x = the number of shots each fires; then, $2x$ = the whole number. Then, $\frac{7x}{12}+\frac{9x}{12}=32$; whence, $x=24$.

18. Let x = the number of acres. Then, $3\times150(x-25)=150x+750$; expanding, $450x-11250=150x+750$; whence, $x=40$.

19. Let x = their money at first;

then, $\frac{3x}{2}-1$ = what A has after first game;

and $\frac{x}{2}+1$ = what B has after first game;

then, $\frac{1}{2}\left(\frac{3x}{2}-1\right)+1$ = what A has after second game;

and $\frac{x}{2}+1+\frac{1}{2}\left(\frac{3x}{2}-1\right)-1$ = what B has after second game.

Then, $\frac{1}{2}\left(\frac{3x}{2}-1\right)+1+2=\frac{x}{2}+1+\frac{1}{2}\left(\frac{3x}{2}-1\right)-1$;

Reducing, $x=6$.

20. Let x = the time to 12 o'clock;

then, $\frac{11}{12}x$ = the space to be gained by minute-hand;

$\frac{11}{12}(x+10)$ = the space to be gained 10 minutes ago.

Then, $\frac{11}{12}x=\frac{2}{3}$ of $\frac{11}{12}(x+10)$;

Reducing, $x=20$.

21. Let x = number of hours;

then, $\dfrac{x}{12}$ = number of revolutions the first makes.

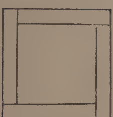
and $\dfrac{x}{16}$ = number of revolutions the second makes.

Then, $\dfrac{x}{12}-\dfrac{x}{16}=1;$

Clearing of fractions, $16x-12x=192;$

whence, $x=48.$

22.



Let x = number of men in front in the 1st formation;

then, $x-4$ = number of men in side of each rectangle;

and $4x-16$ = number of men in each rectangle;

y = number of men in front in the 2d formation;

then, $y-8$ = number of men in side of each rectangle;

and $8y-64$ = number of men in each rectangle.

Then, $x-y=16;$ (1)

and $4(4x-16)=4(8y-64).$ (2)

Reducing (2), $x-2y=-12;$ (3)

Subtracting (1) from (3), $y=28;$

whence, $4(8y-64)=640.$

23. Let x = tens' digit;

and y = units' digit;

Then, $10x+y=4x+4y;$ (1)

and $10x+y+18=10y+x.$ (2)

From (1), $2x=y;$ (3)

From (2), $y-x=2;$ (4)

Substituting (3) in (4), $2x-x=2;$

whence, $x=2;$

$y=4.$

24. Let x = first number;

and $x+a$ = second number;

then, $x+\dfrac{a}{2}$ = arithmetical mean.

Taking dif. of squares, $x^2+2ax+a^2-x^2=2ax+a^2=2a\left(x+\dfrac{a}{2}\right).$

25. Let $x =$ the first number;
and $x+2 =$ the second number;
then, $x+1 =$ the mean.

Difference of squares, $x^2+4x+4-x^2=4x+4$;
Multiplying mean by 4, $=4(x+1)$

26. Let $x =$ tens' digit;
and $y =$ units' digit.

Then, $10x+y+9=x+10y$; (1)
and $10x+y+x+10y=33$; (2)

Reducing (1), $y-x=1$;
Reducing (2), $y+x=3$;
whence, $y=2$;
and $x=1$.

27. Let $x =$ the number;
then, $x-1 =$ the lesser number;
and $x+1 =$ the greater number.

The square $=x^2$;
the product $=(x+1)(x-1)=x^2-1$;
the difference $=x^2-(x^2-1)=1$.

28. Let $x =$ the length;
and $y =$ the breadth.

Then, $(x+3)(y+2)=xy+64$; (1)
and $(x+2)(y+3)=xy+68$. (2)

Expanding (1), $xy+3y+2x+6=xy+64$;
Reducing, $3y+2x=58$; (3)
Expanding (2), $xy+3x+2y+6=xy+68$;
Reducing, $3x+2y=62$; (4)
Multiplying (3) by 2, $6y+4x=116$; (5)
Multiplying (4) by 3, $6y+9x=186$. (6)
Subtracting (5) from (8), $5x=70$;
whence, $x=14$;
and $y=10$.

29. Let $x=$ tens' digit;

and $y=$ units' digit.

Then, $2(10x+y)+36=2(10y+x)-36;$ (1)

and $10x+y=4(x+y)+3.$ (2)

Reducing (1), $y-x=4;$ (3)

Reducing (2), $2x-y=1;$ (4)

Adding (3) and (4) $x=5;$

and $y=9.$

30. Let $x=$ the number of hours A travels;

then, $x-\frac{2}{3}=$ the number of hours B travels.

Since B travels half a mile more than half the distance, and A half a mile less, B travels one mile further than A.

Therefore, $4\frac{1}{2}(x-\frac{2}{3})-3\frac{1}{2}x=1;$

whence, $x-3=1;$

and $x=4.$

Then, $3\frac{1}{2}x+\frac{1}{2}=14\frac{1}{2}$ miles, or half the distance;

whence, the whole distance is 29 miles.

This may also be solved as follows:

Let $x=$ the number of miles A travels;

and $x+1=$ the number of miles B travels.

Then, $\frac{x}{3\frac{1}{2}}=\frac{x+1}{4\frac{1}{2}}+\frac{2}{3};$

whence, $x=14$, and the whole distance is 29 miles.

31. Let $x=$ the distance he walked before returning;

and $y=$ the distance he ran.

Then, $\frac{x-y}{3\frac{1}{2}}=\frac{5}{60}=\frac{1}{12};$ (1)

and $\frac{x}{3\frac{1}{2}}+\frac{y}{7}+\frac{5}{60}=\frac{25}{60}.$ (2)

Clearing (1) of fractions, $2x-2y=\frac{7}{12};$ (3)

Clearing (2) of fractions, $2x+y=\frac{7}{3};$ (4)

Subtracting (3) from (4), $3y=\frac{21}{12};$

whence, $y=\frac{7}{12}.$

This may be worked with one unknown quantity as follows:

Let $x =$ the distance he ran.

As he walked 5 minutes in returning, it must have taken 10 minutes to walk that distance both ways; hence, we have $25-10$, or 15 minutes left for the distance he ran.

Then, $\frac{2x}{7}+\frac{x}{7}=\frac{1}{4}$ of an hour;

whence, $x=\frac{7}{12}$.

32. Let $x =$ the number of miles the 1st goes in 1 second;

and $y =$ the number of miles the 2d goes in 1 second;

then, $92+84=176$ feet $=\frac{1}{30}$ mile, distance gone in passing each other.

Then, $1\frac{1}{2}x+1\frac{1}{2}y=\frac{1}{30}$; (1)

and $6x-6y=\frac{1}{30}$; (2)

Multiplying (1) by 4, $6x+6y=\frac{4}{30}$. (3)

Adding (2) and (3), $x=\frac{1}{72}$;

and $y=\frac{1}{120}$;

then, $3600x = 50$ miles, distance 1st goes in an hour.

and $3600y = 30$ miles, distance 2d goes in an hour.

33. Let $x =$ the first fraction;

and $1-x =$ the second fraction.

First, plus square of second $= x+1-2x+x^2=1-x+x^2$;

Second, plus square of first $=x^2+1-x \quad =1-x+x^2$.

34. Let $8x =$ the length of field; and $5x =$ the breadth of field.

Then, $40x^2=4\times 4\times 40$; whence, $x^2=16$; and $x=\pm 4$; then, $8x=32$; and $5x=20$.

35. Let $x =$ the greater number;

$17-x =$ the lesser number.

Then, $\frac{17-x}{x}:\frac{x}{17-x}::64:81$:

By Theorem I., $81\frac{(17-x)}{x}=\frac{64x}{17-x}$;

Clearing of fractions, $81(17-x)^2=64x^2$;

Extracting square root, $9(17-x)=8x$;

whence, $x=9$;

and $17-x=8$.

36. Let x = the first number;
and y = the second number.

Then, $xy = a$; (1)

and $\frac{x}{y} = b.$ (2)

From (2), $x = by$; (3)
Substituting in (1), $by^2 = a$;

whence, $y = \pm\sqrt{\frac{a}{b}}$;

and $x = \pm\sqrt{ab}.$

37. Let x = the number of children; then, x^2 = the number of books; and $12x^3$ = the cost of books.
Then, $12x^3 = 1500$; whence, $x = 5$.

38. Let $3x$ = the length of field; and $2x$ = the breadth of field; then, $\frac{6x^2}{160}$ = the number of acres.

Then, $\frac{6x^2}{160} \times 3x = 4\frac{1}{2}(10x)$; reducing, $9x^2 = 3600$; whence, $3x = 60$; and $2x = 40$.

39. Let x = A's number;
and y = B's number;

then, $\frac{36}{y}$ = the price of one of A's eggs;

and $\frac{16}{x}$ = the price of one of B's eggs.

Then, $x + y = 100$; (1)

and $\frac{36x}{y} = \frac{16y}{x}.$ (2)

Clearing (2) of fractions, $36x^2 = 16y^2$; (3)
Extracting the square root, $6x = 4y$;
whence, $x = \frac{2}{3}y$;

Substituting in (1), $\frac{2y}{3} + y = 100$;

whence, $y = 60$;
and $x = 40.$

40. Let x = the number of gentlemen; and x^2 = the number of servants; and $4x^2$ = the number of dollars each had.
Then, $4x^2 = 484$; whence, $x = 11$.

41. Let x = the length; and $\frac{4x}{5}$ = the breadth.

Then, $\frac{4}{5}\left(\frac{4x^2}{5}\right)=400$; whence, $x=25$; and $\frac{4x}{5}=20$.

42. Let x = the length of side of smaller; and $x+10$ = the length of side of larger.

Then, $x^2+20x+100 : x^2 :: 25 : 9$; by Theorem I., $9x^2+180x+900=25x^2$; whence, $x=15$; and $x+10=25$.

43. Let x = the number of sheep. Then, $\frac{80}{x}-\frac{80}{x+4}=1$; clearing of fractions, $x^2+4x=320$; whence, $x=16$.

44. Let x = the number of persons. Then, $\frac{110}{x}+1=x$; clearing of fractions, $110+x=x^2$; whence, $x=11$.

45. Let x = the number of yards; then, $\frac{36}{x}$ = the price of one yard.

Then, $4x=\frac{144}{x}+36$: clearing of fractions, $4x^2=144+36x$; whence, $x=12$.

46. Let x = the time it takes first pipe;
and y = the time it takes second pipe.

Then, $$\frac{1}{x}+\frac{1}{y}=\frac{1}{4}; \qquad (1)$$
and $$x-y=6. \qquad (2)$$
From (2), $$x=y+6;$$
Substituting in (1), $$\frac{1}{y+6}+\frac{1}{y}=\frac{1}{4};$$
Clearing of fractions, $$4y+4y+24=y^2+6y;$$
whence, $$y=6;$$
and $$x=12.$$

47. Let x = the length of rectangle;
and y = the breadth.

Then, $$2x+2y=440+4; \qquad (1)$$
and $$xy=12100-4. \qquad (2)$$
Reducing (1), $$x+y=222; \qquad (3)$$
Reducing (2), $$xy=12096; \qquad (4)$$
whence, $$x=126;$$
and $$y=96.$$

48. Let x = the first term;

and y = the common difference.

Then, $x+2y=4x;$ (1)

and $x+5y=17;$ (2)

From (1), $x=\frac{2y}{3};$

Substituting in (2), $\frac{2y}{3}+5y=17;$

whence, $y=3;$

and $x=2.$

Then the first six terms are 2, 5, 8, 11, 14, 17.

49. Let x = the length of fence on one side; and $4x$ = the whole length of fence.

Then, $\frac{x^2}{160}=\frac{4x}{8};$ whence, $x=80;$ and $4x=320.$

50. Let x = the number of scores for half a crown; then, $\frac{30}{x}$ = the price per score.

Then, $\frac{30}{x}-\frac{30}{x+\frac{1}{2}}=3;$ reducing, $3x^2+\frac{3}{2}x=15;$ whence, $x=2;$ and $\frac{30}{x}=15.$

51. Let x = the number of scores for half a crown; then, $\frac{30}{x}$ = the price per score.

Then, $\frac{30}{x-\frac{1}{2}}-\frac{30}{x}=3;$ reducing, $3x^2-\frac{3x}{2}=15;$ whence, $x=2\frac{1}{2};$ and $\frac{30}{x}=12.$

52. Let x = the side of cube. Then, $x^3=4\sqrt{3x^2};$ dividing by x, $x^2=4\sqrt{3};$ whence, $x=\pm 2\sqrt[4]{3}.$

53. Let x = the width of border; and $60+4x$ = the length of border. Then, $60x+4x^2=216;$ whence, $x=3.$

54. Let x = the number of miles a day B traveled; and $x+8$ = the number of miles a day A traveled; then, $\frac{x}{2}$ = the number of days.

Then, $\frac{x^2}{2}+\frac{x^2+8x}{2}=320$; whence, $x=16$; and $\frac{x^2}{2}=128$; and $\frac{x^2+8x}{2}=192$.

55. Let x = A's venture; and y = B's venture.

Then, $\frac{xy}{100}=120$, (1); and $\frac{y^2}{400}=36$, (2).

From (2), $y=120$; substituting in (1), $x=100$.

56. Let x = the number of silver coins; and $27-x$ = the number of copper coins.

Then, $2(27x-x^2)=100$; whence, $x=2$; and $27-x=25$.

57. Let x = the first term;
and y = the common difference.

Then, $x+2y=18$; (1)
and $x+6y=30$. (2)

Subtracting (1) from (2), $4y=12$;
whence, $y=3$;
and $x=12$.

From Formula I., p. 257, $S=\frac{1}{2}n[2a+(n-1)d]=\frac{17}{2}(24+16\times 3)=612$.

58. Let x = the cost of the horse; and $\frac{x^2}{100}$ = the gain.

Then, $x+\frac{x^2}{100}=264$; whence, $x=120$.

59. Let $4x$ = the first number; and $5x$ = the second number.

Then, $4x+10 : 5x-10 :: 5 : 4$; by Theorem I., $16x+40=25x-50$; whence, $x=10$; and $4x=40$; and $5x=50$.

60. Let x = tens' digit; and y = units' digit.

Then, $(10x+y)^2-(10y+x)^2=100x^2+20xy+y^2-100y^2-20xy-x^2$
$=99x^2-99y^2$.

61. Let $x=$ the number of rods built per day.

and $\frac{105}{x}=$ the number of days.

Then, $\frac{105}{x-2}-\frac{105}{x}=6.$

Clearing of fractions, $105x-105x+210=6x^2-12x;$

whence, $x=7.$

62. Let $x=$ the first term;

and $x+2=$ the second term.

Then, by Formula I., $S=[2a+(n-1)d]\frac{n}{2};$

Substituting, $35=[2x+(x+1)2]\frac{x+2}{2};$

Reducing, $x^2+\frac{5}{2}x=\frac{33}{2};$

whence, $x=3.$

63. Let $x=$ the first fraction;

$x-1=$ the second fraction.

First + square of second, $x+x^2-2x+1=x^2-x+1;$

Square of first − second, $x^2-(x-1)=x^2-x+1.$

64. Let $x=$ the width of walk;

and $12+10-x=$ the length of walk.

Then, $22x-x^2=\frac{120}{10}=12;$

whence, $x=11-\sqrt{109}=.5597+\text{rods}=9.234$ feet.

65. Let $x=$ the value of a gold coin;

and $y=$ the value of a silver coin.

Then, $8x+9y=6x+19y;$

whence, $1x=5y;$

By Theorem III., $y:x::1:5.$

66. Let $x=$ the length of picture;

and $y=$ the breadth of picture;

$\frac{120}{20}=$ the number of square feet in picture.

Then, $xy=6;$ (1)

and $2x+2y=10.$ (2)

whence, $x=3;$

and $y=2.$

67. Let x = the distance to the top.

Then, $\frac{x}{2\frac{1}{3}}+\frac{x}{3\frac{1}{2}}=5;$

Clearing of fractions, $5x=35;$

and $x=7;$

and $2x=14$, the whole distance.

68. Let x = the number of B's horses;

and $\frac{18}{x}$ = the rate per horse by first condition;

and $\frac{20}{x+2}$ = the rate per horse by second condition.

Then, $\frac{18}{x}(x+4)=\frac{20}{x+2}(x+6);$

Reducing, $x^2+6x=72;$

whence, $x=6;$

and $\frac{18}{x}(x+4)=30.$

69. Let x = A's stock;

and $x+250$ = B's stock;

$\frac{3x}{3x+5x+1250}$ = A's share of gain.

Then, $\frac{3x}{8x+1250}\times 700=x-300;$

Clearing of fractions, $2100x=8x^2-2400x+1250x-375000;$

whence. $x=500.$

70. Let x = the number of days A travels;

and $x-5$ = the number of days B travels;

then, since $S=(a+l)\frac{n}{2}$, $(1+x)\frac{x}{2}$ = the distance A travels;

and $12(x-5)$ = the distance B travels.

Then, $(1+x)\frac{x}{2}=12(x-5);$

Reducing, $x^2+x=24x-120;$

whence, $x=8$, or 15.

B overtakes A in 8 days from the time A started. But A's rate of traveling is constantly increasing; hence, on the 8th day he travels 8 miles, and on the 15th day, 15 miles. On the 8th day B overtakes A, and on the 15th, A overtakes B.

71. Let x = the number of students;
and y = the fare of each;
then, xy = the expense.

Then, $(x+5)(y+1)=xy+61\frac{1}{2};$ (1)
and $(x-3)(y-1\frac{1}{2})=xy-42.$ (2)

Expanding (1), $xy+5y+x+5=xy+61\frac{1}{2};$
Reducing, $5y+x=56\frac{1}{2};$ (3)
Expanding (2), $xy-3y-1\frac{1}{2}x+4\frac{1}{2}=xy-42;$
Reducing, $3y+1\frac{1}{2}x=46\frac{1}{2};$ (4)
Multiplying (4) by $\frac{2}{3}$, $2y+x=31;$ (5)
Subtracting (5) from (3), $3y=25\frac{1}{2};$
whence, $y=8\frac{1}{2};$
and $x=14.$

72. Let x = the number of days second travels;
and $x+3$ = the number of days first travels;
then, by Formula I., page 257,

$$[2+(x+2)2]\frac{x+3}{2}=\text{the distance first travels.}$$

and $$[24+(x-1)1]\frac{x}{2}=\text{the distance second travels.}$$

Then, $$[2+(x+2)2]\frac{x+3}{2}=(24+x-1)\frac{x}{2}.$$

Expanding, $2x^2+12x+18=24x+x^2-x;$
whence, $x=2$, or $9.$

73. Let x = the distance;
and $\frac{x}{19}$ = the number of days B traveled.

Then, $$\left(\frac{x}{19}\right)^2+7\left(\frac{x}{19}\right)=x-32;$$

Reducing, $$\left(\frac{x}{19}\right)^2-12\left(\frac{x}{19}\right)+36=4;$$

whence, $$\frac{x}{19}=8, \text{ or } 4;$$

and $x=152$, or $76.$

74. Let x = the rate of first train;
and y = the rate of second train;
then, $9x$ = the distance first train goes after meeting;
and $16y$ = the distance second train goes after meeting;
and $9x+16y$ = the whole distance.

Then, $16y-9x=108;$ (1)

and $\dfrac{9x}{y}=\dfrac{16y}{x}.$ (2)

From (2), $9x^2=16y^2$

Extracting square root, $3x=4y;$

whence, $x=\dfrac{4y}{3};$

Substituting in (1), $16y-12y=108;$

whence, $y=27;$

and $x=36;$

and $9x+16y=756.$

This example may also be solved as follows:

Let $x=$ the distance second train ran before meeting;

and $x+108=$ the distance first train ran before meeting;

then, $\dfrac{x+108}{16}=$ the rate of second;

and $\dfrac{x}{9}=$ the rate of first.

Then, $\dfrac{16x}{x+108}=\dfrac{9(x+108)}{x};$

Clearing of fractions, $16x^2=9(x+108)^2;$

Extracting square root, $4x=3x+324:$

whence, $x=324;$

and $x+108=432;$

then, $2x+108=756;$

and $\dfrac{x+108}{16}=27;$

and $\dfrac{x}{9}=36.$

75. Let $x=$ the rate of criminal;

and $x+3=$ the rate of pursuers;

then, $\dfrac{10x}{3}=$ the number of hours required.

Then, $2\frac{2}{5}x+2\frac{2}{5}(x+3)=10x-24;$

Reducing, $26x=156;$

whence, $x=6;$

and $\dfrac{10x}{3}=20.$

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